What they are and how many they are

Master Course, Spring term 2025

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Contents. Basic definitions. Eisenstein series. The discriminant function. Some Fourier expansions. The valence formula and dimension.

1.1 Defining modular forms

Before defining anything, let us spend a couple of paragraphs motivating the concept.

A modular function can be considered a meromorphic function $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^* \longrightarrow \mathbb{C}$. Differential forms are important objects in geometry and holomorphic differential forms play a special role in the theory of Riemann surfaces. The local expression of each of them is $f(z)\,dz$ with f holomorphic and under a change of chart $z \mapsto g(z)$ it becomes $f\big(g(z)\big)g'(z)\,dz$. The map $z \mapsto \gamma z$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ leaves $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^*$ invariant and $\frac{d}{dz}\gamma z = j_\gamma^{-2}(z)$ with j_γ as in (2). Then we are led to the condition

$$j_\gamma^{-2}(z)f(\gamma z) = f(z)$$

and we should add something specifying that the differential form is also holomorphic at infinity. The factor $j_\gamma^{-2}(z)$ is replaced by $j_\gamma^{-k}(z)$ with $k \in \mathbb{Z}_{\ge 0}$ even when one considers tensor products of differential forms.

A more elementary (albeit less informative) motivation is to consider homogeneous functions F from the set of lattices to $\mathbb{C}$. If for each lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ we express these functions in terms of the generators ω_1 and ω_2, the homogeneity condition reads

$$F(\omega_1, \omega_2) = \omega_2^p F(\omega_1/\omega_2, 1).$$

We have already seen that under the normalization $\Im(\omega_1/\omega_2) > 0$ two lattices are equal if and only if their generators are related by an element of $\mathrm{SL}_2(\mathbb{Z})$. The previous formula applied to the identity $F(\omega_1, \omega_2) = F(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$ gives

$$f(z) = j_\gamma^p(z) f(\gamma z) \qquad \text{taking} \quad f(z) = F(z, 1).$$

If p is not integer then we shall find problems to preserve the cocycle condition of Lemma 3.2 because it is not possible to fix a principal value for general $z_1, z_2 \in \mathbb{C} - \mathbb{R}$ such that $(z_1z_2)^p = z_1^p z_2^p$ for $p \notin \mathbb{Z}$. If $p \in \mathbb{Z}^+$ then $j_\gamma^p(z) \to \infty$ as $z \to i\infty$ for $\gamma \ne \pm T^n$ and it could ruin the behavior of f at infinity. On the other hand, $j_S(z) = -j_{S^3}(z)$ implies that p odd can only occur for $f = 0$. This leads again to $j_\gamma^{-k}(z)$ with $k \in \mathbb{Z}_{\ge 0}$ even.

After this motivation, we define a *modular form* of *weight* $k \in \mathbb{Z}_{\ge 0}$ as a function $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying

$$f(z) = j_\gamma^{-k}(z) f(\gamma z) \qquad \text{for every} \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}) \tag{1}$$

that is holomorphic in $\mathbb{H}$ at also at infinity (as a 1-periodic function). According to previous considerations about modular functions, the latter means that there is a converging Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n \, e^{2\pi i n z} \qquad \text{for} \quad z \in \mathbb{H}. \tag{2}$$

The Fourier expansion makes sense because (1) with $\gamma = T$ proves that f is 1- periodic.

If $a_0 = 0$ we say that f is a *cusp form*. They are relevant because they remain bounded at any cusp when approaching it vertically, not only at $\mathfrak{a} = i\infty$. This is not the case for the rest of modular forms. For instance, note that putting $\gamma = S$ in (1) it follows $f(z) = z^{-k} f(-1/z)$ and if $a_0 \ne 0$ then f behaves as $a_0 z^{-k}$ when $z = it$, $t \in \mathbb{R}^+$, tends to the cusp $\mathfrak{b} = 0$.

The set of modular forms of weight k clearly defines a linear space that is called $\mathcal{M}_k$. On the other hand, $\mathcal{S}_k$ denotes the subspace of cusp forms. A holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying (1), but perhaps not being homorphic at infinity (not having a Fourier expansion as before), is said to be a *weakly modular form* of weight k. Note that the previous remark $j_S(z) = -j_{S^3}(z)$ shows that $f = 0$ is the only weakly modular form of odd weight. In particular, $\mathcal{M}_k = \mathcal{S}_k = \{0\}$ for k odd.

It can be proved that modular forms are the same as weakly modular forms with a subexponential growth as $\Im(z) \to \infty$ [9] and this growth is also equivalent to the boundedness at all cusps meaning that $j_\gamma^{-k}(z) f(\gamma z)$ is bounded as $\Im(z) \to \infty$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We do not enter in these equivalences here.

The coefficients in the Laurent series of $\wp$ (Lemma 4.4) give the first examples of weakly modular forms. We define for $k \in \mathbb{Z}_{>2}$ even the *Eisenstein series* and the *normalized Eisenstein series* of weight k, respectively, as

$$G_k(z) = \sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} (mz+n)^{-k} \qquad \text{and} \qquad E_k(z) = \frac{1}{2\zeta(k)} G_k(z).$$

Recall that $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$, a special value of the *Riemann zeta function*. Note that these definitions for odd k would give identically zero functions.

Lemma 1.1. *For $k \in \mathbb{Z}_{>2}$ even G_k and E_k are weakly modular functions of weight k.*

Proof. Of course, it is enough to prove it for G_k. As mentioned when we introduced the $\wp$ function, for any lattice $\sum_{\omega \in \Lambda^*} |\omega|^{-p} < \infty$ when $p > 2$ and standard results in complex analysis [1, 5.1.1] show that G_k defines an holomorphic function $\mathbb{H} \longrightarrow \mathbb{C}$.

For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we can write $m\gamma z + n$ as

$$(m,\, n) \begin{pmatrix} \gamma z \\ 1 \end{pmatrix} = j_\gamma^{-1}(z)\,(m,\, n) \begin{pmatrix} j_\gamma(z)\gamma z \\ j_\gamma(z) \end{pmatrix} = j_\gamma(z)\,(m,\, n)\gamma \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

Since $(m,\, n) \mapsto (m',\, n') = (m,\, n)\gamma$ defines a bijection in $\mathbb{Z}^2 - \{(0,0)\}$, because γ is invertible, it follows $G_k(z) = j_\gamma^k(z) G_k(z)$. □

The reason to introduce the normalizing factor in E_k is to have $E_k(i\infty) = 1$. This is clear if we separate the terms with $m = 0$ and use the symmetry $(m,n) \leftrightarrow (-m,-n)$. For later reference, we state the result of these manipulations.

Lemma 1.2. *For $k \in \mathbb{Z}_{>2}$ even the Eisenstein series admit the alternative series representations*

$$G_k(z) = 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}(mz+n)^{-k} \qquad \text{and} \qquad E_k(z) = 1 + \sum_{\substack{m=1 \\ \gcd(m,n)=1}}^{\infty}\sum_{n\in\mathbb{Z}}(mz+n)^{-k}.$$

After the previous comments, only the coprimality condition in the second formula could require some thought (Exercise 1).

1.2 When Eisenstein met Fourier

To prove that the Eisenstein series are actually modular forms, it remains to show that they have a Fourier expansion of the form (2). We are going to write it in terms of

$$B_k = -\frac{2\zeta(k)\, k!}{(2\pi i)^k} \qquad \text{where } k \in \mathbb{Z}^+ \text{ is even.}$$

This is known to be a rational number, in fact, it is the k-th *Bernoulli number* defined as the k-th derivative at 0 of $x/(e^x - 1)$ (see [3, Th. 1.4] for a proof). Here we take it as a mere abbreviation. This is a small table of values (see A027641 and A027642 in [6] for a longer one):

k	4	6	8	10	12	14	16	18
B_k	$-1/30$	$1/42$	$-1/30$	$5/66$	$-691/2730$	$7/6$	$-3617/510$	$43867/798$
$-2k/B_k$	240	-504	480	-264	$65520/691$	-24	$16320/3617$	$-28728/43867$

The last row is motivated by the next result.

Proposition 1.3. *For $k \in \mathbb{Z}_{>2}$ even and any $z \in \mathbb{H}$*

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\, e^{2\pi i n z} \qquad \text{where} \quad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$$

and only positive divisors d of n are considered.

It is easy to see that this expansion converges because the exponential decay of $e^{2\pi i n z}$ kills the polynomial growth of $\sigma_{k-1}(n)$. So, we have

Corollary 1.4. *For k as before, $G_k, E_k \in \mathcal{M}_k - \mathcal{S}_k$.*

For the proof of Proposition 1.3 we need an analytic auxiliary result.

Lemma 1.5. *For any $z \in \mathbb{H}$*

$$\sum_{n\in\mathbb{Z}} (z+n)^{-2} = -4\pi^2 \sum_{\ell=1}^{\infty} \ell\, e^{2\pi i \ell z}$$

with absolute convergence.

Proof. Without entering into details, the absolute convergence follows from the convergence of $\sum_{n=1}^{\infty} n^{-2}$ and $\sum_{n=1}^{\infty} n\, e^{-\varepsilon n}$ for $\varepsilon > 0$.

Consider the function $f(x) = \pi e^{-2\pi i r x}$ in $[-1/2, 1/2]$ where $r \in \mathbb{R} - \mathbb{Z}$. A rather trivial computation shows that the Fourier coefficients of its 1-periodic extension are $c_n = \frac{(-1)^n}{r+n} \sin(\pi r)$. Applying Parseval identity ($f \in L^2$ and $\|f\|_2 = \pi$) it is deduced

$$\pi^2 = \sin^2(\pi r) \sum_{n\in\mathbb{Z}} (r+n)^{-2}.$$

By analytic continuation and the absolute convergence this is also true for r in $\mathbb{C} - \mathbb{Z}$. Take $r = z \in \mathbb{H}$ to get, using $2i \sin x = e^{ix} - e^{-ix}$,

$$\sum_{n\in\mathbb{Z}} (z+n)^{-2} = \frac{\pi^2}{\sin^2(\pi z)} = -\frac{4\pi^2 e^{2\pi i z}}{\left(1 - e^{2\pi i z}\right)^2}$$

and the result follows from the Taylor expansion $(1-w)^{-2} = 1 + 2w + 3w^2 + 4w^3 + \dots$ having radius of convergence 1. ☐

Proof of Proposition 1.3. Differentiating $k-2$ times in Lemma 1.5, renaming z as mz and summing $m \in \mathbb{Z}^+$

$$(k-1)! \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} (mz+n)^{-k} = -4\pi^2 (2\pi i)^{k-2} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i \ell m z} = (2\pi i)^k \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e^{2\pi i n z}.$$

The second equality essentially reduces to note that ℓ divides $n = \ell m$. Substituting in the first formula of Lemma 1.2 and dividing by $2\zeta(k)$, the proof is complete. ☐

1.3 Modular discrimination

Substituting the values $B_4 = -1/30$ and $B_6 = 1/42$ [8] it is deduced that the Fourier coefficients of E_4 and E_6 are integers. Namely, Proposition 1.3 gives

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi i n z} \qquad \text{and} \qquad E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, e^{2\pi i n z}.$$

We are going to use them to construct our first cusp form, the *discriminant function*

$$\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728}.$$

It is very important in the theory and closely related to arguably the most influential work of Ramanujan [7] in which he considered some arithmetical properties of its Fourier coefficients.

We highlight first a couple of results although their proofs reduce to a dull calculation.

Proposition 1.6. *We have $\Delta \in \mathcal{S}_{12}$ and $a_1 = 1$ in its Fourier expansion.*

Proof. Let us write $q = e^{2\pi i z}$ and $O(q^n)$ to indicate a tail series composed by terms of order at least n.

Since $\sigma_3(1) = \sigma_5(1) = 1$, we have $E_4(z) = 1 + 240q + O(q^2)$ and $E_6(z) = 1 - 504q + O(q^2)$ that imply $E_4^3(z) = 1 + 720q + O(q^2)$ and $E_6^2(z) = 1 - 1008q + O(q^2)$ giving $a_0 = 0$ and $a_1 = 1$ in the Fourier expansion of the discriminant function. □

The name "discriminant" is not arbitrary. If we recall the motivation to define the j function, the next result shows that it is actually a discriminant.

Proposition 1.7. *We have the equality $j(z) = E_4^3(z)/\Delta(z)$ and $\Delta(z) \neq 0$ in $\mathbb{H}$.*

Proof. We have $G_4(z) = 2\zeta(4)E_4(z) = \frac{\pi^4}{90}E_4(z)$ and $G_6(z) = 2\zeta(6)E_6(z) = \frac{\pi^6}{945}E_6(z)$. Substituting in (4) and simplifying the coefficients, $j(z) = 1728\, E_4^3(z)/\big(E_4^3(z) - E_6^2(z)\big)$ is obtained. By the definition of j, the denominator in this expression is a constant multiple of $g_2^3 - 27g_3^2$ for $\Lambda = z\mathbb{Z} + \mathbb{Z}$. It does not vanish by Lemma 4.7. □

Now we are ready to deduce a slightly stronger form of Theorem 4.13.

Corollary 1.8. *The function j is a modular function with a pole of order 1 at infinity, $a_{-1} = 1$ and $a_n \in \mathbb{Q}$ for $n \in \mathbb{Z}_{\geq -1}$.*

To deduce this corollary from Proposition 1.7, we are implicitly dividing power series in $q = e^{2\pi i z}$ using the information $a_1 = 1$ from Proposition 1.6. To clarify this point and for later reference, let us review the structure of the division of two power series.

Lemma 1.9. *If $f(z) = \sum_{n=0}^{\infty} b_n z^n$ and $g(z) = \sum_{n=0}^{\infty} c_n z^n$ with $c_0 = 1$ define holomorphic functions for $|z| < 1$ and $g(z) \neq 0$, then $f(z)/g(z) = \sum_{n=0}^{\infty} d_n z^n$ for $|z| < 1$ with $d_n = b_n - \sum_{j=1}^{n} c_j d_{n-j}$. In particular, by repeated use of this recurrence, d_n is a polynomial with integer coefficients in $b_0, \dots, b_n, c_0, \dots, c_n$.*

The proof reduces to write $\sum_{n=0}^{\infty} c_n z^n \cdot \sum_{n=0}^{\infty} d_n z^n = \sum_{n=0}^{\infty} b_n z^n$ and compare the coefficients of the same degree.

In the theory of elliptic functions or of modular functions, the apparent freedom to construct many examples is an illusion and causes surprising identities. This rigidity of the examples also occurs for modular forms and it will be theorized in the next subsection. Now, the following product expansion for the discriminant is a good example.

Theorem 1.10. *We have*

$$\Delta(z) = P(z) \qquad \text{where} \quad P(z) = e^{2\pi i z} \sum_{n=1}^{\infty} \big(1 - e^{2\pi i n z}\big)^{24}.$$

In particular, the Fourier coefficients of Δ are integers.

Note that the convergence of P is assured by the exponential decay to 0 of $|e^{2\pi i n z}|$.

The important point is that once one knows that P shares with Δ the property of being a modular form of weight 12 (which is not easy to prove), to deduce the identity is plain sailing.

Proposition 1.11. *We have $P \in \mathcal{M}_{12}$ for P as before.*

The key to deduce $\Delta = P$ and other unexpected relations between modular forms relies on proving that there are very few. Let us emphasize it stating with the new notation something already noticed for modular functions.

Lemma 1.12. $\mathcal{M}_0 = \mathbb{C}$.

Proof. Any $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ in $\mathcal{M}_0$ is identically $a_0 = \lim_{\Im(z)\to\infty} f(z)$ by Proposition 4.14. □

Proof of Theorem 1.10. The Fourier expansion of P has $a_0 = 0$ and $a_1 = 1$ as that of Δ. Using the nonvanishing in Proposition 1.7 and Lemma 1.9 (with $z \mapsto e^{2\pi i z}$), $f(z) = P(z)/\Delta(z)$ is holomorphic and admits a Fourier expansion $\sum_{n=0}^{\infty} d_n e^{2\pi i n z}$ with $d_0 = 1$, in particular, $\lim_{\Im(z)\to\infty} f(z) = 1$. Proposition 1.11 and Proposition 1.6 show $f \in \mathcal{M}_0$ and Lemma 1.12 concludes that f is identically $\lim_{\Im(z)\to\infty} f(z) = 1$. □

Before proving Proposition 1.11, let us use Theorem 1.10 to introduce some historical comments. The Fourier coefficients of Δ define the so-called *Ramanujan τ function.* Namely, $\tau : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ is the arithmetic function satisfying

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)\, e^{2\pi i n z}.$$

Theorem 1.10 gives $\tau(1) = 1$, $\tau(2) = -24$ and $\tau(3) = 252$ quite quickly. Computing higher values, for instance $\tau(10)$, require a big effort to be done by hand. Using alternative formulas and his phenomenal computational abilities, Ramanujan included in [7] a table with the first 30 values of $\tau(n)$ (and probably he knew a longer one). With the computed values he obtained experimentally rules to compute $\tau(n)$ in terms of $\tau(p)$ for p in the factorization of n. In particular, he observed that τ is *multiplicative*:

$$\tau(mn) = \tau(m)\tau(n) \qquad \text{if } \gcd(m,n) = 1.$$

For instance, from $\tau(2) = -24$ and $\tau(3) = 252$ it follows $\tau(6) = -6048$. This seems very mysterious and Ramanujan did not prove it. The first proof was provided shortly after by Mordell using heavily the connection with elliptic function. Two decades later, Hecke found a way to show that this was a general phenomenon in the theory of modular forms when they are conveniently normalized. Note for instance that the σ_{k-1} appearing in the Fourier expansion of the Eisenstein series are also multiplicative functions.

Ramanujan proved that τ allows to write a formula for the number of representations as a sum of 24 squares. For instance, for $p > 2$ prime, it reads

$$r_{24}(p) = \frac{16}{691} p^{11} + \frac{33152}{691}\tau(p) + \frac{16}{691}.$$

With this formula at hand, it is natural to ask about the size of $\tau(p)$. One hopes $\tau(p)$ to be very small in comparison to the main term p^{11}. The bound claimed for $\tau(p)$ and extended to other cusp forms under the name *Ramanujan–Petersson conjecture*, turned to be a very hard and important problem related to the so-called *Riemann hypothesis over finite fields*. It was solved in Deligne in 1971 who got the Fields Medal in 1978.

In the proof of Proposition 1.11, we shall appeal to two auxiliary results.

Lemma 1.13. *For $x > 0$*

$$\sum_{n=1}^{\infty} \log\left(1 - e^{-nx}\right) = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{\coth(kx) - 1}{k}.$$

Proof. Using the Taylor expansion of $\log(1-x)$ that has radius of convergence 1, the left hand side is

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} k^{-1}e^{-2nkx} = -\sum_{k=1}^{\infty} k^{-1}\sum_{n=1}^{\infty} e^{-2nkx} = -\sum_{k=1}^{\infty} \frac{k^{-1}e^{-2kx}}{1 - e^{-2kx}}$$

and it is plain that $2e^{-2t}/(1 - e^{-2t}) + 1 = \coth t$. □

Lemma 1.14. *The residues of $f(z) = \pi^2\cot(\pi z)/\sin^2(\pi iz/t)$ are*

$$\operatorname{Res}(f,0) = \frac{\pi}{3}(t^2+1), \qquad \operatorname{Res}(f,k) = -\frac{\pi}{\sinh^2(\pi k/t)}, \qquad \operatorname{Res}(f,ikt) = -\frac{\pi t^2}{\sinh^2(\pi kt)}$$

where $k \in \mathbb{Z} - \{0\}$.

The proof is a calculation (Exercise 4).

Proof of Proposition 1.11. The Fourier expansion comes from the very definition and it must be proved that P is weakly modular of weight 12. It is obvious that $P(z) = P(Tz) = j_T^{-12}(z)P(Tz)$ and it remains to show $P(-1/z) = z^{12}P(z)$ because $j_S^{-12}(z) = z^{-12}$ (Theorem 1.1 and Lemma 3.2 are implicit here).

By the principle of analytic continuation, it is enough to prove it for $z = it$ with $t > 0$. Taking logarithms, the relation becomes the real identity

$$-\frac{2\pi}{t} + 24\sum_{n=1}^{\infty} \log\left(1 - e^{-2\pi n/t}\right) = 12\log t - 2\pi t + 24\sum_{n=1}^{\infty} \log\left(1 - e^{-2\pi nt}\right).$$

By Lemma 1.13 this is equivalent to

$$\log t = \frac{\pi}{6}\left(t - t^{-1}\right) + \sum_{k=1}^{\infty} \frac{1}{k}\left(\coth(\pi kt) - \coth(\pi k/t)\right).$$

This is true for $t = 1$, then it follows if the derivatives of both sides multiplied by $2t^2$ coincide, which reads (using that $\sinh^2 x$ is even)

$$2t = \frac{\pi}{3}\left(t^2+1\right) - \pi\sum_{k\in\mathbb{Z}-\{0\}} \left(\frac{t^2}{\sinh^2(\pi kt)} + \frac{1}{\sinh^2(\pi k/t)}\right).$$

By Lemma 1.14 and the residue theorem, this is

$$2t = \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\partial\mathcal{R}_N} f \qquad \text{with} \quad \mathcal{R}_N = \Big\{z\in\mathbb{C} \,:\, |\Re(z)| + \frac{|\Im(z)|}{t} \le N+\frac{1}{2}\Big\}$$

where $N\in\mathbb{Z}^+$. Note that $\mathcal{R}_N$ is a rhombus of vertices $\pm(N+1/2)$ and $\pm it(N+1/2)$. The function f is odd and $\cot(\pi z)\to -i$ on the part of the boundary in $\mathbb{H}$. Then the limit of the integral is

$$2\lim_{N\to\infty}\int_{\partial\mathcal{R}_N\cap\mathbb{H}} \frac{-i\pi\,dz}{\sin^2(\pi i z/t)} = \lim_{N\to\infty}\frac{-2i\pi t}{\tanh(\pi z/t)}\Big|_{z=N+\frac{1}{2}}^{N+\frac{1}{2}} = 4\pi i t$$

because $\tanh(\pm\infty)=\pm 1$ and the proof is complete. □

1.4 Valence without chemistry

We have already shown in Lemma 1.12 that there are not nontrivial modular forms of weight 0. The question is how many modular form there are for higher weights and how many of them are cusp forms. As the underlying structure is of linear space, the right question to ask is about the dimension or a basis.

The answer is complete with the following results:

Theorem 1.15. *For each $k\ge 2$ even*

$$\dim\mathcal{M}_k = \begin{cases}\lfloor k/12\rfloor & \text{if } 12\mid k-2\\ 1+\lfloor k/12\rfloor & \text{if } 12\nmid k-2,\end{cases} \qquad \dim\mathcal{S}_k = \begin{cases}0 & \text{if } k<12\\ \dim\mathcal{M}_k - 1 & \text{if } k\ge 12.\end{cases}$$

Theorem 1.16. *A basis of $\mathcal{M}_k$ for $k\ge 0$ even is $\big\{E_4^jE_6^\ell \,:\, 4j+6\ell=k \text{ with } j,l\in\mathbb{Z}_{\ge 0}\big\}$ and a basis of $\mathcal{S}_k$ for $k\ge 12$ even is $\big\{E_4^jE_6^\ell\Delta \,:\, 4j+6\ell=k-12 \text{ with } j,l\in\mathbb{Z}_{\ge 0}\big\}$.*

If we have a basis, the natural approach in linear algebra is to get the dimension counting elements, but we are going to proceed the other way around deducing Theorem 1.16 as a corollary of Theorem 1.15 with some extra considerations.

Both results are a consequence of the so-called *valence formula*, a kind of very poor version of *Riemann-Roch theorem* that can be proved with elementary complex analysis.

We write $n(f,z_0)\in\mathbb{Z}$ for the *order* of a meromorphic function f at z_0. This means that the limit of $f(z)(z-z_0)^{-n(f,z_0)}$ when $z\to z_0$ is different from 0 and ∞. We already introduced the concept for 1-periodic functions meromorphic at infinity. Consequently, we write $n(f,\infty)=-N$, or $n(f,i\infty)=-N$, if the Fourier expansion is

$$f(z) = \sum_{n=-N}^{\infty} a_n\, e^{2\pi i n z} \qquad \text{with} \quad a_{-N}\ne 0.$$

If f is a weakly modular function (even admitting meromorphic functions) then $n(f,\gamma z_0) = n(f,z_0)$ for any $z_0\in\mathbb{H}$ and completing this with the convention $n(f,\gamma\infty)=n(f,\infty)$, the map $z\mapsto n(f,z)$ is well defined on the orbits $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^*$. In the following statement we keep the notation z for the orbits.

Theorem 1.17 (valence formula). *Any non identically zero modular form f of weight k satisfies*

$$\sum_{z\in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{H}^*} w_z n(f,z) = \frac{k}{12}$$

where w_z is $1/2$ and $1/3$, respectively, for the orbits of i and $e^{2\pi i/3}$ and $w_z = 1$ otherwise.

The proof shows that it also works when f is meromorphic (even at infinity) and for $k \in \mathbb{Z}$, not necessarily positive [5]. For $z \in \mathbb{H}$, the inverse of w_z is the cardinality of the stability group of z in $\mathrm{PSL}_2(\mathbb{Z})$.

Let us start with a result not depending on the valence formula.

Lemma 1.18. *If $k \ge 12$ is even then*

$$\mathcal{S}_k = \{f\Delta \,:\, f \in \mathcal{M}_{k-12}\} \qquad \text{and} \qquad \mathcal{M}_k = \mathcal{S}_k \oplus \langle E_k\rangle.$$

In particular, $\dim \mathcal{S}_k = \dim \mathcal{M}_{k-12}$ *and* $\dim \mathcal{M}_k = \dim \mathcal{M}_{k-12} + 1$.

Proof. As $\Delta(z) \ne 0$ for $z \in \mathbb{H}$ and $n(\Delta,\infty) = 1$, for any $g \in \mathcal{S}_k$ the function $f = g/\Delta$ is holomorphic in $\mathbb{H}$ and at infinity and it is weakly modular of weight $k-12$. It proves $\mathcal{S}_k \subset \{f\Delta \,:\, f \in \mathcal{M}_{k-12}\}$. The reversed inclusion is trivial.

If $f \in \mathcal{M}_k$ has an expansion $f(z) = \sum_{n=0}^\infty a_n e^{2\pi i n z}$ then $f - a_0 E_k \in \mathcal{S}_k$, showing the equality $\mathcal{M}_k = \mathcal{S}_k \oplus \langle E_k\rangle$.

The dimension relations are immediate consequences. □

Proof of Theorem 1.15. By Lemma 1.18 it is enough to prove the result for $0 \le k < 12$ even because $\lfloor k/12\rfloor$ increases by 1 when k increases by 12.

For a cusp form $n(f,\infty) \ge 1$ and Theorem 1.17 gives a contradiction for $k \le 12$ and $\mathcal{S}_k = \{0\}$ follows. Hence $\dim \mathcal{M}_k \le 1$ because otherwise a linear combination of two modular forms would give a cusp form. The Eisenstein series prove that $\dim \mathcal{M}_k = 1$ for $k \ge 4$. Lemma 1.12 gives $\dim \mathcal{M}_0 = 1$ and the only remaining case is $k = 2$. Multiplying by 6 the valence formula with $k = 2$ it is obtained a sum of even numbers adding 1. This contradiction shows $\mathcal{M}_2 = \{0\}$. □

Our proof of the bases for $\mathcal{M}_k$ and $\mathcal{S}_k$ employs an elegant argument taken from [2].

Proof of Theorem 1.16. By Lemma 1.18 it is enough to show the result for $\mathcal{M}_k$.

Solving Bézout's equation, $4j+6l = k$ gives $j = -\frac{k}{2}+3m$, $\ell = \frac{k}{2}-2m$ with $m \in [k/6, k/4]\cap\mathbb{Z}$ to assure the nonnegativity. It is not difficult to see that the cardinality of this set and the formula for $\dim \mathcal{M}_k$ coincide (Exercise 5).

It remains to prove that $E_4^j E_6^\ell$ are linearly independent. If

$$\sum_{k/6\le m\le k/4} a_m E_4^{-k/2+3m} E_6^{k/2-2m} = 0$$

is a nontrivial linear combination, multiplying by $(E_4/E_6)^{k/2}$ and substituting $E_4^3/E_6^2 = j/(j-1728)$, it follows that $P(j) = 0$ for a nonzero polynomial P. It is impossible because $j(z) \to \infty$ when $z \to i\infty$. □

Proof of Theorem 1.17. Consider the compact region $\mathcal{R}$ given by $\overline{\mathcal{F}}$ omitting small neighborhoods of the three elliptic points in $\partial\mathcal{F}$ and of $i\infty$. Namely,

$$\mathcal{R} = \overline{\mathcal{F}} - \bigcup_{z_0\in E} \{|z-z_0|<r\} - \{|\Im(z)|>r^{-1}\} \qquad \text{with} \quad E = \{e^{2\pi i/3}, 1+e^{2\pi i/3}, i\}$$

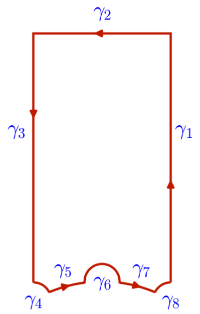
and $r>0$ a small parameter. Assume that f does not vanishes on $\partial\mathcal{R}$ (at the end of the proof we mention the modifications if this is not the case).

According to the argument principle [1] and recalling that the orbits of the points in $\mathcal{F}$ are distinct, for r small enough,

$$\sum_{z\in\{i\infty,i,e^{2\pi i/3}\}} w_z n(f,z) + \int_{\partial\mathcal{R}} g = \sum_{z\in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^*} w_z n(f,z) \qquad \text{with} \quad g = \frac{f'}{2\pi i f}.$$

The boundary of $\mathcal{R}$ with the positive orientation can be written as the union of 8 oriented circular or linear paths $\{\gamma_j\}_{j=1}^8$ that we start numbering from the right vertical segment.

$$\partial\mathcal{R} = \bigcup_{j=1}^{8}\gamma_j$$



$$\begin{aligned}
\gamma_1 &= \text{segment of } \left\{\Re(z)=\tfrac{1}{2}\right\}\\
\gamma_2 &= \text{segment of } \left\{\Im(z)=r^{-1}\right\}\\
\gamma_3 &= \text{segment of } \left\{\Re(z)=-\tfrac{1}{2}\right\}\\
\gamma_4 &= \text{arc of } \left\{|z-e^{2\pi i/3}|=r\right\}\\
\gamma_5 &= \text{arc of } \left\{|z|=1\right\}\\
\gamma_6 &= \text{arc of } \left\{|z-i|=r\right\}\\
\gamma_7 &= \text{arc of } \left\{|z|=1\right\}\\
\gamma_8 &= \text{arc of } \left\{|z-e^{2\pi i/3}-1|=r\right\}
\end{aligned}$$

By the 1-periodicity, $\int_{\gamma_1\cup\gamma_3} g = 0$ and the valence formula follows proving that the four quantities

$$(3) \qquad n(f,i\infty)+\int_{\gamma_2} g, \qquad -\frac{k}{12}+\int_{\gamma_5\cup\gamma_7} g, \qquad \frac{n(f,e^{2\pi i/3})}{3}+\int_{\gamma_4\cup\gamma_8} g, \qquad \frac{n(f,i)}{2}+\int_{\gamma_6} g$$

go to zero when $r\to 0^+$.

The Fourier expansion of f is of the form $\sum_{n=n(f,i\infty)}^{\infty} a_n\, e^{2\pi i n z}$ with $a_{n(f,i\infty)}\neq 0$. Then g has a Fourier expansion of the form $n(f,i\infty)+\sum_{n=1}^{\infty} b_n\, e^{2\pi i n z}$ and the first quantity in (3) is identically zero.

The modularity implies $f(-1/z)=z^k f(z)$, hence $f'(-1/z)=kz^{k-1}f(z)+z^k f'(z)$. Dividing both equations it is obtained the functional equation

$$z^{-2}g(-1/z) = \frac{k}{2\pi i z} + g(z).$$

Using that $z\mapsto Sz=-1/z$ preserves $\gamma_5\cup\gamma_7$ except for changing the orientation,

$$-\int_{\gamma_5\cup\gamma_7} g = \int_{\gamma_5\cup\gamma_7} g(-1/z)\, d(-1/z) = \frac{k}{2\pi i}\int_{\gamma_5\cup\gamma_7}\frac{dz}{z} + \int_{\gamma_5\cup\gamma_7} g.$$

When $r \to 0^+$ the last but one integral gives i times the variation of the angle on $|z| = 1$ (just parametrize $z = \cos t + i \sin t$) from $e^{2\pi i/3}$ to $e^{2\pi i/3} + 1$, which is $-\pi/6$. Then the second quantity in (3) tends to 0.

For any $z_0 \in \mathbb{H}$ the Taylor expansion $f(z) = \sum_{n=n(f,z_0)}^{\infty} a_n(z - z_0)^n$ implies that g differs from $\frac{1}{2\pi i} n(f, z_0)(z - z_0)^{-1}$ in a holomorphic function in a neighborhood of z_0. Then

$$\lim_{r\to 0^+} \int_{\gamma_4 \cup \gamma_8} g = \frac{n(f, e^{2\pi i/3})}{2\pi i} \lim_{r\to 0^+} \int_{\gamma_4} \frac{dz}{z - e^{2\pi i/3}} + \frac{n(f, e^{2\pi i/3} + 1)}{2\pi i} \lim_{r\to 0^+} \int_{\gamma_8} \frac{dz}{z - e^{2\pi i/3} - 1}.$$

The integrals over γ_4 and γ_8 are $-i\pi/3$ considering the variation of the angle and $n(f, e^{2\pi i/3}) = n(f, e^{2\pi i/3} + 1)$ by the periodicity. It proves that the third quantity in (3) goes to 0. The same argument applies for the last one using that $\int_{\gamma_6} \frac{dz}{z-i} \to -i\pi$ because γ_6 becomes closer and closer to a semicircle.

Let us address now the issue of the possible zeros of f on $\partial\mathcal{R}$. As they cannot have an accumulation point (even infinity) they can be avoided on the "shrinking" paths γ_2, γ_4, γ_6 and γ_8 taking r small enough. A zero $z_0 = \frac{1}{2} + it$ on γ_1 has its companion $z_0' = -\frac{1}{2} + it$ on γ_3. Changing $\mathcal{R}$ by $\mathcal{R} \cup \{|z - z_0| \le \varepsilon\} - \{|z - z_0'| < \varepsilon\}$ the contribution of the orbit of z_0 is not double counted and the argument works in the same way. Finally, a zero z_0 on γ_5 is paired to a zero $z_0' = Sz_0$ on γ_7. Considering a closed hyperbolic disk D of radius ε centered at z_0, SD is a hyperbolic disk of the same radius around z_0', because S is an isometry. Changing $\mathcal{R}$ by $\mathcal{R} \cup D - \mathrm{Int}(SD)$, the new path replacing $\gamma_5 \cup \gamma_7$ is still invariant by S and the functional equation for g can be employed in the same way. □

The exact determination of bases and dimensions allow to obtain some exotic formulas. For instance, it is clear that $E_4^2, E_8 \in \mathcal{M}_8$ and both have $a_0 = 1$ as their first Fourier coefficient. The formula in Theorem 1.15 implies $\mathcal{M}_{12} = 1$, hence $E_4^2 = E_8$. Comparing the Fourier expansions (Proposition 1.3), it follows this strange relation between divisor functions (Exercise 8)

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m)$$

Using that $\dim \mathcal{M}_{10}$ is also 1, one concludes in the same way $E_4E_6 = E_{10}$. We know that Δ generates $\mathcal{S}_{12}$ and $E_6^2 - E_{12} \in \mathcal{S}_{12}$ implies that both functions are proportional, because $\dim \mathcal{S}_{12} = 1$. Working out the value of a Fourier coefficient it is deduced a formula for τ in terms of divisors [4, (1.65)].

A last example with higher dimension is $\mathcal{M}_{16}$. We have $E_{16} \in \mathcal{M}_{16}$ and $\{E_4E_6^2, E_4^4\}$ is a basis of this space by Theorem 1.16. Hence $E_{16} = \lambda_1 E_4 E_6^2 + \lambda_2 E_4^4$ for some $\lambda_1, \lambda_2 \in \mathbb{Q}$ that can be determined using the two first Fourier coefficients of E_4 and E_6.

Exercises

Exercise 1. Prove Lemma 1.2.

Exercise 2. Fill the details to assure that the Fourier expansion of E_k is actually convergent for every $z \in \mathbb{H}$.

Exercise 3. Explain why the derivative of $j(z)$ is a weakly modular form but it is not a modular form.

Exercise 4. Check Lemma 1.14

Exercise 5. Write an elementary argument showing that for $k/2 \in \mathbb{Z}^+$ the number of non-negative integer solutions of $4x + 6y = k$ and the formula for $\dim \mathcal{M}_k$ coincide.

Exercise 6. Show with a theoretical argument (with numerical calculations doable with a scientific calculator) that $E_4\big(\frac{22+4i}{25}\big)$ differs from $(3+4i)^4$ in a number of absolute value less than $2 \cdot 10^{-6}$.

Exercise 7. Show that there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $E_{12} = \lambda_1 E_6^2 + \lambda_2 \Delta$ and compute them knowing that $B_6 = \frac{1}{42}$ and $B_{12} = \frac{691}{2730}$.

Exercise 8. Use $E_4^2 = E_8$ to deduce the identity

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m)$$

and check it numerically for $n = 2, 3, 4$.

Exercise 9. Let $f = 7E_4^3 + 5E_6^2$. Show that there exist $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that $E_{12} = \lambda_1 f + \lambda_2 \Delta$ and compute them. Deduce from it that 691 divides $p^{11} + 1 - \tau(p)$ for any prime number p. This is due to Ramanujan.

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