Primer parcial 27/Feb/2019 Variable Compleja II

Nombre y apellidos..

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1) Decide si las siguientes afirmaciones son verdaderas o falsas incluyendo en cada caso una pequeña justificación.

a) [1 punto] La serie de Taylor de $f(z) = z^{12}/(z^6+64)$ en $z_0 = 0$ tiene radio de convergencia igual a 1.

b) [1.5 puntos] La función $20/\big(\wp^{(4)}(z) + 19\big)$ es una función racional de $\wp(z)$.

c) [1.5 puntos] El residuo de $3\wp'(z)\Big(\dfrac{\operatorname{sen} z}{z}\Big)^{2019}$ en $z = 0$ es 2019.

2) [3 puntos] Demuestra

$$\sum_{n\in\mathbb{Z}}(-1)^n 5^{-2n^2} = \prod_{n=1}^{\infty}\big(1 + 5^{1-2n}\big)\big(1 - 5^{-n}\big).$$

Indicación. Utiliza la fórmula de triple producto de Jacobi con $q = 1/\sqrt{5}$.

3) [3 puntos] Demuestra que si una función entera cumple $|f(z)| < |z|^2 + 1$ para todo $z \in \mathbb{C}$ y $f''(0) = 0$, entonces es una función lineal $f(z) = az + b$.

Fórmulas sobre funciones elípticas

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\Big(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\Big) = \frac{1}{z^2} + \sum_{k=1}^{\infty} a_{2k} z^{2k} \qquad \text{con} \qquad a_{2k} = (2k+1)\sum_{\omega\in\Lambda^*}\omega^{-2k-2}$$

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \qquad \text{con} \qquad g_2 = 60\sum_{\omega\in\Lambda^*}\omega^{-4}, \quad g_3 = 140\sum_{\omega\in\Lambda^*}\omega^{-6}$$

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i n z} = \prod_{n=1}^{\infty}\big(1 - q^{2n}\big)\big(1 + q^{2n-1}e^{2\pi i z}\big)\big(1 + q^{2n-1}e^{-2\pi i z}\big) \quad \text{con} \quad q = e^{\pi i \tau}, \quad \Im\tau > 0$$

$$\theta(z+\tau) = q^{-1}e^{-2\pi i z}\theta(z) \qquad \text{y} \qquad \wp(z;1,\tau) = A_\tau \frac{\theta^2(z+1/2)}{e^{2\pi i z}\theta^2(z+\tau^*)} + B_\tau = -\Big(\frac{\theta'(z+\tau^*)}{\theta(z+\tau^*)}\Big)' + C_\tau.$$

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Name ....

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1) Decide whether the following claims are true or false writing in each case a brief justification

a) [1 point] The radius of convergence of the Taylor series of $f(z) = z^{12}/(z^6+64)$ at $z_0 = 0$ equals 1.

b) [1.5 points] The function $20/\big(\wp^{(4)}(z)+19\big)$ can be expressed as a rational function on $\wp(z)$.

c) [1.5 points] The residue of $3\wp'(z)\Big(\dfrac{\operatorname{sen} z}{z}\Big)^{2019}$ at $z=0$ is 2019.

2) [3 points] Prove the identity

$$\sum_{n\in\mathbb{Z}}(-1)^n 5^{-2n^2} = \prod_{n=1}^{\infty}\big(1+5^{1-2n}\big)\big(1-5^{-n}\big).$$

Hint. Use Jacobi triple product formula with $q = 1/\sqrt{5}$.

3) [3 points] Prove that if an entire function satisfies $|f(z)| < |z|^2+1$ for every $z\in\mathbb{C}$ and $f''(0)=0$ then it is a linear function $f(z) = az+b$.

Some formulas on elliptic functions

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\Big(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\Big) = \frac{1}{z^2} + \sum_{k=1}^{\infty} a_{2k} z^{2k} \qquad \text{with} \qquad a_{2k} = (2k+1)\sum_{\omega\in\Lambda^*}\omega^{-2k-2}$$

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \qquad \text{with} \qquad g_2 = 60\sum_{\omega\in\Lambda^*}\omega^{-4}, \quad g_3 = 140\sum_{\omega\in\Lambda^*}\omega^{-6}$$

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i n z} = \prod_{n=1}^{\infty}\big(1-q^{2n}\big)\big(1+q^{2n-1}e^{2\pi i z}\big)\big(1+q^{2n-1}e^{-2\pi i z}\big) \quad \text{with} \quad q = e^{\pi i \tau}, \quad \Im\tau > 0$$

$$\theta(z+\tau) = q^{-1}e^{-2\pi i z}\theta(z) \qquad \text{and} \qquad \wp(z;1,\tau) = A_\tau \frac{\theta^2(z+1/2)}{e^{2\pi i z}\theta^2(z+\tau^*)} + B_\tau = -\Big(\frac{\theta'(z+\tau^*)}{\theta(z+\tau^*)}\Big)' + C_\tau.$$
