Smoothing Minimally Supported Frequency (MSF) Wavelets : Part I

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Abstract

All orthonormal wavelets ψ for which $\hat{\psi}$ has support contained in $S_\alpha = [\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$, $0 < \alpha < 2\pi$, are characterized. It is shown that all these wavelets are associated with multiresolution analyses. Some properties of the topology of the set of orthonormal wavelets as a subset of the unit sphere in $L^2(\mathbb{R})$ are obtained.

1 Introduction

In this paper we continue the study of orthonormal wavelets based on the properties of their Fourier transforms. An orthonormal wavelet is defined to be a function $\psi \in L^2(\mathbb{R})$ such that $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where

$$\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j x - k), \qquad j, k \in \mathbb{Z}.$$

In this paper we shall use the word wavelet to refer to an orthonormal wavelet. It turns out that ψ, with $||\psi||_2 = 1$, is such a function if and only if the following two basic equations are satisfied:

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1, \qquad a.e.\ \xi \in \mathbb{R}; \tag{1.1}$$

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\,\overline{\hat{\psi}(2^j(\xi + 2k\pi))} = 0, \quad a.e.\ \xi \in \mathbb{R}, \quad \text{for each} \quad k \in 2\mathbb{Z}+1. \tag{1.2}$$

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That this simple (but by no means easy) characterization of all orthonormal wavelets is, indeed, true was announced in several places. [Le1] appears to be, chronologically, the first publication containing an announcement of this result; however, P.G. Lemarié cites some previous lectures of Y. Meyer as containing this statement. Proof of this fact, in the general case, can be found in [Gri] and a different proof is presented in [Wan]. For many purposes it is advantageous to add two additional equations to these two:

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+2k\pi)|^2 = 1, \qquad a.e.\ \xi \in \mathbb{R}; \tag{1.3}$$

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(2^j(\xi+2k\pi))\,\overline{\hat{\psi}(\xi+2k\pi)} = 0, \qquad a.e.\ \xi \in \mathbb{R}, \quad \text{for each} \quad j \geq 1. \tag{1.4}$$

One of the reasons for including these last two equations is that they characterize the orthonormality of the system $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$. It is also seen more directly that the first two equations characterize the completeness of the system. At first sight, it is somewhat surprising that they also characterize the orthonormality. However, this was observed by several authors (see [Dau] and [BSW]). When ψ is band-limited (i.e. $\hat{\psi}$ has bounded support) it is easier to establish the validity and the meaning of these four equations (see [BSW]). A second reason for including the last two equations is that we shall make repeated use of them in our proofs.

A considerable number of results in the wavelet literature are based on these four equations. It is our feeling, however, that their implications are far from having been exhausted. This is the basis for our statement at the beginning of this introduction. We are continuing the project, began in [BSW], of understanding the various propeties of orthonormal wavelets based on equations (1.1), (1.2), (1.3) and (1.4) satisfied by their Fourier transforms.

In order to be precise we use the following definition of Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx.$$

We will be consistent in our use of Greek letters $\xi, \eta, \cdots$ to denote the "frequency" variable and Roman letters $x, y, \cdots$ to denote the "space" (or "time") variable. The best known and oldest example of an orthonormal wavelet is the Haar wavelet

$$h(x) = \begin{cases} 1, & \text{if } 0 \leq x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

This wavelet (practically the only one known for about 70 years) is the "prototype" of the compactly suported wavelets whose smoother versions were introduced by I.

Daubechies (see [Dau] and the references given there). In this paper our principal interest is focused on those wavelets that are connected to the ones that are band-limited. The "simplest" one of this class, playing a rôle similar to that which $h(x)$ plays among the compactly supported wavelets, is the Shannon wavelet:

$$\psi_S(x) = 2\,\mathrm{sinc}(2\pi x) - \mathrm{sinc}(\pi x) = \frac{\sin(2\pi x) - \sin(\pi x)}{\pi x}$$

whose Fourier transform is

$$\hat{\psi}_S(\xi) = \chi_I(\xi),$$

where $I = [-2\pi, -\pi] \cup [\pi, 2\pi]$. P.G. Lemarié and Y. Meyer ([LM]) constructed corresponding wavelets that are smooth in the frequency variable as well. More precisely, their wavelets ψ have Fourier transforms that are supported in $[-\frac{8}{3}\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \frac{8}{3}\pi]$ that can be chosen to be in the class C^∞ and with even absolute values. This class was completely characterized in [BSW]. In particular, in this paper, it was shown that this support is "most natural".

In order to consider a class that places these wavelets in a more general context, let us observe that equation (1.3) has as an immediate consequence the fact that, for any wavelet ψ, the support of $\hat{\psi}$ must have measure that is at least 2π. Moreover, this minimal measure is attained only if $|\hat{\psi}(\xi)|$ equals 0 or 1 almost everywhere. We shall call such wavelets Minimally Supported Frequency (MSF) wavelets.

A natural question that arises from this definition is to characterize all subsets $K \subset \mathbb{R}$ (which must be of measure 2π) that are the supports of the Fourier tranforms of MSF wavelets. This problem, considered in [FW] and [HKLS], is not the central one here. Our main concern is to consider specific (and usually, simple) supporting sets and show how to obtain "smoother" versions of the MSF wavelets associated with K (much in the same sense that the Lemarié-Meyer wavelets are smooth versions of the Shannon wavelet). Some of our results are, at first glance, a bit surprising. It is natural to conjecture that by enlarging K only a little bit one can obtain smoother wavelets (in the frequency variable), particulary if K is a simple union of only a few intervals.

This is not the case, however. This follows from recent results of P. Auscher ([Aus]) and P.G. Lemarié ([Le2] and [Le3]) that show that if the Fourier transform of a wavelet has continuous absolute value and a "mild" decrease at infinity, then the wavelet must arise from a Multiresolution Analysis (MRA) (we shall make these statements precise below). But there exists an MSF wavelet discovered by J.L. Journé,

$$\hat{\psi}(\xi) = \chi_K, \qquad K = [-\tfrac{32}{7}\pi, -4\pi] \cup [-\pi, -\tfrac{4}{7}\pi] \cup [\tfrac{4}{7}\pi, \pi] \cup [4\pi, \tfrac{32}{7}\pi],$$

that cannot arise from an MRA. Moreover, if this is the case, we shall prove that nearby smooth wavelets must also have this last property.

Our goal is to construct new wavelets from certain specific MSF wavelets. These "new" wavelets can be smooth; however, they can also exhibit (in the Fourier transform side) "Cantor like" discontinuities.

We shall begin our study by giving a complete characterization of those wavelets "arising" from the Shannon wavelet. Let us be more precise: we have already presented a special class of these wavelets, the Lemarié-Meyer wavelets, where the Fourier transform has support in the interval $[-\frac{8}{3}\pi, \frac{8}{3}\pi]$. We shall completely characterize **all** the wavelets whose support for the Fourier transform is contained in this interval. Perhaps the most startling result we obtain in this connection is that, given any measurable function b of absolute value not exceeding 1, supported in $[\frac{2}{3}\pi, \frac{4}{3}\pi]$, all such wavelets have a Fourier transform that can be easily expressed on the rest of $[-\frac{8}{3}\pi, \frac{8}{3}\pi]$ in terms of b. In fact we shall characterize a more general class of wavelets associated with MSF wavelets that have Fourier transforms supported on sets that are not necessarily symmetric about the origin. The Fourier transforms of these wavelets are supported on $[-\frac{8}{3}a, 4\pi - \frac{4}{3}a]$ or on $[-4\pi + \frac{4}{3}a, \frac{8}{3}a]$, $0 < a \leq \pi$. This will be done in section 2. The rôle played by the Shannon wavelet here is assumed by the MSF wavelet ψ_α, whose Fourier transform is

$$\hat{\psi}_\alpha = \chi_{[-2\alpha,-\alpha]\cup[2\pi-\alpha,4\pi-2\alpha]}, \qquad 0 < \alpha < 2\pi.$$

It is, perhaps, appropriate to indicate why ψ_α is, indeed, a wavelet. It was observed by [FW] and [HKLS] that the sets K of measure 2π, that are the support of the Fourier transform of a wavelet ψ are completely characterized by equations (1.1) and (1.3). These are equivalent to the statement: the translates by $2\ell\pi$, $\ell \in \mathbb{Z}$, of K form a partition of $\mathbb{R}$, and the same is true for the dilates by 2^j, $j \in \mathbb{Z}$, of K. From this it is clear that ψ_α is an MSF wavelet.

The third section will be devoted to some questions that arise naturally when considering the notion of an MRA. We assume that the reader is familiar with the construction of the wavelet from an MRA, though we will find it necessary to give a brief description of this procedure (both [Dau] and [Mey] give complete treatments of the properties of an MRA). We shall show that all the wavelets constructed in section 2 arise from an MRA. In addition, we shall show that wavelets that are not associated with an MRA cannot be approximated (in L^2) by wavelets that are.

A special case of theorem 3.10, as well as the sufficiency in theorem 2.1, have been recently announced, without detailed proofs, in [Han]. This author does not announce, however, the characterization of all wavelets whose Fourier transform has support contained in S_α, which is one of the principal features of our paper. We have become aware of this article while we were in the final stages of our typing process.

In a forthcoming paper we shall consider the problem of smoothing MRA-wavelets by obtaining smoothing versions of their associated low-pass filters.

2 Wavelets associated with the Shannon wavelet

As stated in the introduction, we shall study wavelets that are associated with the MSF wavelets ψ_α, $0 < \alpha < 2\pi$. Observe that if $\alpha = \pi$, ψ_α is the Shannon wavelet. The support of $\hat{\psi}_\alpha$ is $[-2\alpha, -\alpha] \cup [2\pi - \alpha, 4\pi - 2\alpha]$. If we let $\beta = 2\pi - \alpha$, this set can be written as $[-4\pi + 2\beta, -2\pi + \beta] \cup [\beta, 2\beta]$. This exhibits an obvious symmetry between the classes $\{\psi_\alpha : 0 < \alpha \leq \pi\}$ and $\{\psi_\beta : 0 < \beta \leq \pi\}$ that allows us to study the latter in terms of the former. In this connection, observe that ψ is an orthonormal wavelet if and only if $\tilde{\psi}(x) = \psi(-x)$ is an orthonormal wavelet. For these reasons we can restrict our attention to the case $0 < \alpha \leq \pi$.

We shall prove the following:

Theorem 2.1 *Suppose $\psi \in L^2(\mathbb{R})$ and $b = |\hat{\psi}|$ has support contained in*

$$S_\alpha = [-\tfrac{8}{3}\alpha, 4\pi - \tfrac{4}{3}\alpha], \qquad 0 < \alpha \leq \pi.$$

Then ψ is an orthonormal wavelet if and only if

(i) $b^2(\xi) + b^2(\frac{1}{2}\xi) = 1$, *for a.e.* $\xi \in [4\pi - \frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$;

(ii) $b(\xi) = 1$, *for a.e.* $\xi \in [2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha]$;

(iii) $b^2(\xi) + b^2(\xi + 2\pi) = 1$, *for a.e.* $\xi \in [-\frac{4}{3}\alpha, -\frac{2}{3}\alpha]$;

(iv) $b(\xi) = b(\frac{1}{2}\xi + 2\pi)$, *for a.e.* $\xi \in [-\frac{8}{3}\alpha, -\frac{4}{3}\alpha]$;

(v) $\hat{\psi}(\xi) = e^{ip(\xi)}b(\xi)$, *with* $p(\xi)$ *satisfying*

$$p(\xi) + p(2(\xi - 2\pi)) - p(2\xi) - p(\xi - 2\pi) = (2n(\xi) + 1)\pi,$$

for a.e. $\xi \in D_\alpha \cap (\mathrm{Supp}\, b) \cap (\frac{1}{2}\mathrm{Supp}\, b)$,

where $D_\alpha = [2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha]$, *and* $n(\xi)$ *is an integer-valued measurable function*;

(vi) $b(\xi) = 0$, *for a.e.* $\xi \in [-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha] = H_\alpha$.

In the proof of this theorem, as well as that of proposition (2.3) below, it will be helpful to look at figure 1 which portrays the various intervals considered in the above result.

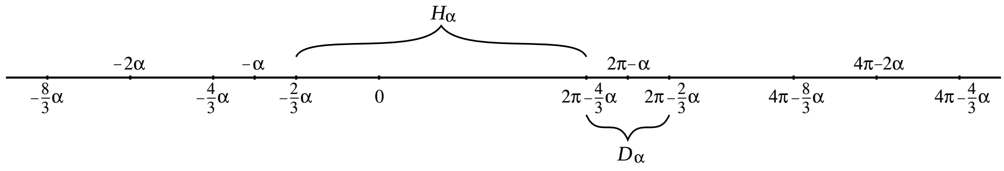

Figure 1

Remark 1. Conditions (i), (iii) and (iv) can be written in the form

$$\begin{aligned} &\text{(a)} \qquad b(2\xi) = \sqrt{1-b^2(\xi)} = b(\xi - 2\pi), \\ &\text{(b)} \qquad b(2(\xi-2\pi)) = b(\xi), \end{aligned} \qquad \text{for } \; a.e.\ \xi \in D_\alpha. \tag{2.2}$$

From this it is particularly easy to see that, if b is any measurable function on D_α with values bewteen 0 and 1, then the wavelet in question is completely determined by (2.2), (ii), (vi) and any choice of the phase function $p(\xi)$ satisfying (v). Figure 2 is helpful for understanding this feature of our theorem.

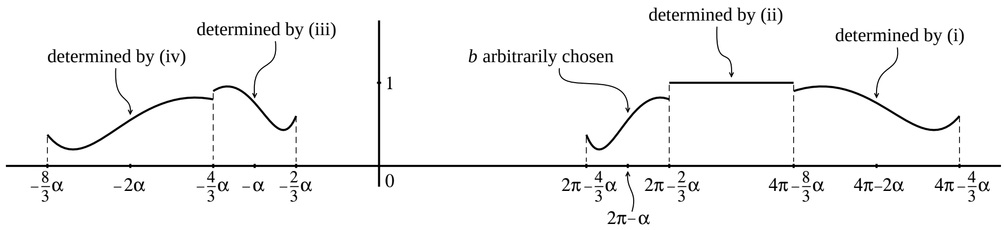

Figure 2

Remark 2. The wavelet ψ is, as stated above,

associated with the MSF wavelet ψ_α. We have already mentioned that there are special rôles played by the intervals

$$[-\tfrac{8}{3}\alpha, -\tfrac{4}{3}\alpha],\ [-\tfrac{4}{3}\alpha, -\tfrac{2}{3}\alpha],\ [2\pi - \tfrac{4}{3}\alpha, 2\pi - \tfrac{2}{3}\alpha], \ \text{ and } \ [4\pi - \tfrac{8}{3}\alpha, 4\pi - \tfrac{4}{3}\alpha].$$

The dyadic dilation structure that is exhibited in these intervals is forced upon us by the "dilation equation" (1.1). These considerations easily lead us to the natural hypothesis that $0 < \alpha \leq \pi$.

Remark 3. When $\alpha = \pi$, the interval $[2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha]$ is trivial. In this case our theorem has much in common with theorem (3.1) in [BSW] that characterizes the Lemarié-Meyer wavelets; however, the present theorem is more general, not only because it includes all positive α less than π, but it involves wavelets for which $|\hat{\psi}| = b$ is not even. Furthermore, we have no smoothness restriction on b (which, as stated above, can be chosen to be a Cantor-like function on D_α).

Remark 4. Observe that the MSF wavelets ψ_α converge (for example, in $L^2(\mathbb{R})$) as $\alpha \to 0$ to the function ψ_H such that $\hat{\psi}_H = \chi_{[2\pi,4\pi]}$. Of course, ψ_H is not a wavelet for $L^2(\mathbb{R})$, but it does generate an orthonormal basis for the classical Hardy space $H = H^2$. It is known that such wavelets cannot have a continuous b (see [BSW] and [Aus]). Our theorem does give us simpler "quasi-progressive" wavelets that the ones found by A. Cohen ([Coh]). These are wavelets for which the Fourier transform has arbitrarily small support in the negative reals and is arbitrarily smooth.

Remark 5. Concerning property (v), it is useful to observe that $(\operatorname{Supp} b) \cap (\frac{1}{2}\operatorname{Supp} b)$ can be a set of measure zero (in fact, this is the situation for all MSF wavelets) or it can be as large as $[-\frac{4}{3}\alpha, -\frac{2}{3}\alpha] \cup D_\alpha$.

We present the proof in three basic steps. First we show that the wavelets in question have Fourier transforms whose support are complementary to the interval H_α.

Proposition 2.3 *Let ψ be an orthonormal wavelet for which* $\operatorname{Supp}\hat{\psi} \subset S_\alpha$, $0 < \alpha \le \pi$, *then $\hat{\psi}(\xi) = 0$ almost everywhere on H_α.*

Proof : Let $J_\alpha = [-\frac{2}{3}\alpha, 2\pi - \frac{2}{3}\alpha] = D_\alpha \cup H_\alpha$. Define

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2.$$

If J is any interval of lenght 2π, then

$$\int_J D_\psi(\xi)\, d\xi = 2\pi. \tag{2.4}$$

This follows from the fact that $\|\hat{\psi}\|_2^2 = 2\pi\|\psi\|_2^2$ and a simple calculation (see the proof of theorem 3.11). We claim that

$$D_\psi(\xi) = 1 - \sum_{j=-\infty}^{0} |\hat{\psi}(2^j\xi)|^2, \qquad \text{for } a.e.\ \xi \in H_\alpha. \tag{2.5}$$

Moreover,

$$D_\psi(\xi) = 1 - |\hat{\psi}(2(\xi - \pi))|^2, \qquad \text{for } a.e.\ \xi \in [2\pi - \tfrac{4}{3}\alpha, 2\pi - \tfrac{2}{3}\alpha] = D_\alpha. \tag{2.6}$$

From this we can deduce that $0 \leq D_\psi(\xi) \leq 1$ for *a.e.* $\xi \in J_\alpha$, and, from (2.4), we conclude that $D_\psi(\xi) = 1$ *a.e.* on J_α. In particular, from (2.5) we see that $\hat{\psi}(\xi) = \hat{\psi}(2^o\xi) = 0$ *a.e.* on H_α, which is the desired result.

Thus, all we need is to establish (2.5) and (2.6). Observe that for almost every $\xi \in H_\alpha$, $2^j(\xi + 2k\pi) \notin S_\alpha$ if $k \neq 0$ and $j \geq 1$. This and (1.1) then give us

$$D_\psi(\xi) = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2 = 1 - \sum_{j=-\infty}^{0} |\hat{\psi}(2^j\xi)|^2, \qquad \text{for } \ a.e.\ \xi \in H_\alpha.$$

This establishes equation (2.5). It is easily verified that, for almost every $\xi \in D_\alpha$, $2^j(\xi + 2k\pi) \notin S_\alpha$ if $j \geq 2$, and $2(\xi + 2k\pi) \notin S_\alpha$ if $k > 0$ or $k < -1$. Consequently,

$$D_\psi(\xi) = |\hat{\psi}(2\xi)|^2 + |\hat{\psi}(2(\xi - 2\pi))|^2, \qquad \text{for } \ a.e.\ \xi \in D_\alpha. \tag{2.7}$$

If we write (1.3) with ξ replaced by 2ξ, we have

$$\sum_{k \in \mathbb{Z}} |\hat{\psi}(2(\xi + k\pi))|^2 = 1, \qquad a.e.\ \xi \in \mathbb{R}.$$

When $\xi \in D_\alpha$, only the terms associated with $k = -2, -1, 0$ do not necessarily vanish. Thus, this last equality becomes

$$|\hat{\psi}(2\xi)|^2 + |\hat{\psi}(2\xi - 2\pi)|^2 + |\hat{\psi}(2\xi - 4\pi)|^2 = 1, \qquad \text{for } \ a.e.\ \xi \in D_\alpha.$$

This and (2.7) give us (2.6).

□

We now begin the proof of theorem 2.1. Let us assume that ψ is an orthonormal wavelet with $\hat{\psi}$ supported in S_α. From proposition 2.3 we now know that $\operatorname{Supp}\hat{\psi} \subseteq \overline{S_\alpha - H_\alpha}$. Equality (i) is a consequence of (1.1) since this equality reduces to

$$b^2(\xi) + b^2(2\xi) = 1, \qquad \text{for } \ a.e.\ \xi \in [2\pi - \tfrac{4}{3}\alpha, 2\pi - \tfrac{2}{3}\alpha]$$

(for the remaining j, $2^j\xi$ lies outside the support of b). Consider the function on the left of (1.3) on the period interval $(2\pi - \frac{4}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$. For ξ in this interval, $\xi + 2k\pi$ lies to the right of the support of $\hat{\psi}$ when $k \geq 1$, and to the left if $k \leq -3$. Hence

$$b^2(\xi - 4\pi) + b^2(\xi - 2\pi) + b^2(\xi) = 1, \quad \text{for } \ a.e.\ \xi \in [2\pi - \tfrac{4}{3}\alpha, 4\pi - \tfrac{4}{3}\alpha]. \tag{2.8}$$

In the subinterval D_α, we also have $b(\xi - 4\pi) = 0$. Thus, equation (1.3) becomes

$$b^2(\xi - 2\pi) + b^2(\xi) = 1, \quad a.e. \ \text{ on } \ D_\alpha.$$

Letting $\eta = \xi - 2\pi$, this coincides with equality (iii). On $(2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha)$, $\xi - 2\pi \in H_\alpha$ and $\xi - 4\pi \leq -\frac{8}{3}\alpha$. Thus, $b^2(\xi) = 1$ for *a.e.* $\xi \in [2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha]$. But this is equality (ii). When $\xi \in (4\pi - \frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha)$, $\xi - 2\pi \in H_\alpha$ and, thus, (2.8) becomes

$$b^2(\xi - 4\pi) + b^2(\xi) = 1, \qquad \text{for } \textit{a.e. } \xi \in [4\pi - \tfrac{8}{3}\alpha, 4\pi - \tfrac{4}{3}\alpha]. \tag{2.9}$$

By letting $\eta = \frac{1}{2}\xi$, we obtain that $b^2(2(\eta - 2\pi)) + b^2(2\eta) = 1$ for almost every $\eta \in [2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha]$. Together with (i) (which has been already proved) this gives us $b(\eta) = b(2(\eta - 2\pi))$ for *a.e.* η in this interval. Writing this in terms of $\xi = 2(\eta - 2\pi)$, we obtain (iv).

Finally, we must establish (v). For this purpose we consider (1.4) with $j = 1$:

$$\sum_{\ell \in \mathbb{Z}} \hat{\psi}(2(\xi + 2\ell\pi))\, \overline{\hat{\psi}(\xi + 2\ell\pi)} = 0, \qquad \text{for } \textit{a.e. } \xi \in \mathbb{R}.$$

When $\xi \in [2\pi - \frac{4}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$, $2(\xi + 2\ell\pi)$ lies outside the support of $\hat{\psi}$ when $\ell < -1$ and when $\ell > 0$. Thus, the last equality has the form

$$\hat{\psi}(\xi)\, \overline{\hat{\psi}(2\xi)} + \hat{\psi}(\xi - 2\pi)\, \overline{\hat{\psi}(2(\xi - 2\pi))} = 0, \quad \textit{a.e. } \xi \in [2\pi - \tfrac{4}{3}\alpha, 4\pi - \tfrac{4}{3}\alpha]. \tag{2.10}$$

In particular, (2.10) shows that, for almost every $\xi \in D_\alpha \subset [2\pi - \frac{4}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$, the two vectors $(\hat{\psi}(\xi), \hat{\psi}(\xi - 2\pi))$ and $(\hat{\psi}(2\xi), \hat{\psi}(2(\xi - 2\pi))$ are orthogonal to each other.

Equality (iii) of the theorem (which has been already proved) tells us that the first vector is normal for *a.e.* $\xi \in D_\alpha$; equality (2.9) gives us the normality for the second vector. Hence, if $\hat{\psi}(\xi) = e^{ip(\xi)} b(\xi)$, we must have

$$\begin{aligned} &e^{i\delta(\xi)} (e^{ip(\xi)} b(\xi),\ e^{ip(\xi - 2\pi)} b(\xi - 2\pi)) \\ &= (-b(2(\xi - 2\pi)) e^{-ip(2(\xi - 2\pi))},\ b(2\xi) e^{-ip(2\xi)}), \qquad \textit{a.e. } \xi \in D_\alpha, \end{aligned} \tag{2.11}$$

for an appropriate real valued measurable function δ. But (2.2) (b) tells us that $b(2(\xi - 2\pi)) = b(\xi)$ *a.e.* on D_α, and (2.2) (a) tells us that $b(\xi - 2\pi) = b(2\xi)$ *a.e.* on D_α. This allows us to rewrite (2.11) in the form

$$\left\{ \begin{array}{l} [e^{i\delta(\xi)} e^{ip(\xi)} + e^{-ip(2(\xi - 2\pi))}]\, b(\xi) = 0, \\ [e^{i\delta(\xi)} e^{ip(\xi - 2\pi)} - e^{-ip(2\xi)}]\, b(2\xi) = 0, \end{array} \right. \qquad for\ \textit{a.e. } \xi \in D_\alpha.$$

It follows that

$$\left\{ \begin{array}{ll} e^{i[\pi + p(\xi) + p(2(\xi - 2\pi))]} = e^{-i\delta(\xi)}, & \text{for } \textit{a.e. } \xi \in D_\alpha \cap (\operatorname{Supp} b), \\ e^{i[p(\xi - 2\pi) + p(2\xi)]} = e^{-i\delta(\xi)}, & \text{for } \textit{a.e. } \xi \in D_\alpha \cap (\tfrac{1}{2}\operatorname{Supp} b). \end{array} \right.$$

Hence, for almost every $\xi \in D_\alpha \cap (\operatorname{Supp} b) \cap (\frac{1}{2}\operatorname{Supp} b)$,

$$p(\xi) + p(2(\xi - 2\pi)) - p(\xi - 2\pi) - p(2\xi) = (2n(\xi) + 1)\pi,$$

for some integer-valued measurable function n. But this is the desired condition (v).

We now turn our attention to the converse and assume ψ satisfies the six conditions stated in the theorem. We will show that equalities (1.1) and (1.2) are satisfied. Though these characterize orthonormal wavelets, including the fact that $\|\psi\|_2 = 1$, this last equality is an

inmediate consequence of the six conditions:

$$2\pi \int_{\mathbb{R}} |\psi|^2 = \int_{S_\alpha} |\hat{\psi}|^2 = \int_{-\frac{8}{3}\alpha}^{-\frac{4}{3}\alpha} b^2 + \int_{-\frac{4}{3}\alpha}^{-\frac{2}{3}\alpha} b^2 + \int_{H_\alpha} 0 + \int_{D_\alpha} b^2 + \int_{2\pi-\frac{2}{3}\alpha}^{4\pi-\frac{8}{3}\alpha} 1 + \int_{4\pi-\frac{8}{3}\alpha}^{4\pi-\frac{4}{3}\alpha} b^2.$$

Changing variables we can express the first, second and sixth terms as integrals over D_α giving us (by (iv), (iii) and (i))

$$\begin{aligned} 2\pi \int_{\mathbb{R}} |\psi|^2 &= \int_{-\frac{8}{3}\alpha}^{-\frac{4}{3}\alpha} b^2(\tfrac{\xi}{2} + 2\pi)\, d\xi + \int_{-\frac{4}{3}\alpha}^{-\frac{2}{3}\alpha} [1 - b^2(\xi + 2\pi)]\, d\xi \\ &\qquad + \int_{D_\alpha} b^2(\xi)\, d\xi + (2\pi - 2\alpha) + \int_{4\pi-\frac{8}{3}\alpha}^{4\pi-\frac{4}{3}\alpha} [1 - b^2(\tfrac{\xi}{2})]\, d\xi \\ &= 2\int_{D_\alpha} b^2 + \tfrac{2}{3}\alpha - \int_{D_\alpha} b^2 + \int_{D_\alpha} b^2 + 2\pi - 2\alpha + \tfrac{4}{3}\alpha - 2\int_{D_\alpha} b^2 = 2\pi. \end{aligned}$$

We now show how to obtain (1.1). Let

$$R(\xi) = \sum_{j\in\mathbb{Z}} b^2(2^j\xi),$$

which is a finite sum for every ξ. On $I_\alpha = (2\pi - \frac{4}{3}\alpha, 4\pi - \frac{8}{3}\alpha]$, $R(\xi) = b^2(\xi) + b^2(2\xi)$ for almost every ξ. By (2.2) (a) this equals 1 almost everywhere on D_α. If $\xi \in [2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha]$, 2ξ lies to the right of S_α and $b(\xi) = 1$ almost everywhere by (ii). This shows that $R(\xi) = 1$ for almost every $\xi \in I_\alpha$. Since $(0,\infty)$ equals the disjoint union of $2^j I_\alpha$, $j \in \mathbb{Z}$, and $R(2^{j_o}\xi) = R(\xi)$ for any $j_o \in \mathbb{Z}$, it follows that $R(\xi) = 1$ for *a.e.* $\xi > 0$. To show that this equality holds for $\xi < 0$ we use the disjoint union $(-\infty, 0) = \bigcup_{j\in\mathbb{Z}} 2^j[-\frac{4}{3}\alpha, -\frac{2}{3}\alpha)$. The rest of the argument is analogous to the one just given. Thus, (1.1) is established.

To obtain (1.2) we consider

$$U_\ell(\xi) = \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\, \overline{\hat{\psi}(2^j(\xi + (2\ell+1)2\pi)}, \qquad \ell \in \mathbb{Z},$$

which is a finite sum for every $\xi \in \mathbb{R}$. We want to show that $U_\ell(\xi) = 0$ almost everywhere. This is clearly true if $\ell \geq 1$ or $\ell \leq -2$ since in either case $2^j(\xi+(2\ell+1)2\pi)$ lies outside S_α if $2^j\xi \in \operatorname{Supp}\hat{\psi}$. Thus, we only need to show that $U_{-1}(\xi) = 0 = U_o(\xi)$ almost everywhere. The fact that $U_o(\xi) = \overline{U_{-1}(\xi + 2\pi)}$ reduces the problem to the study of $U_{-1}(\xi)$. If $\xi \notin D_\alpha$ each summand in the definition of $U_{-1}(\xi)$ must be zero. For $\xi \in D_\alpha$ we have

$$\begin{aligned} U_{-1}(\xi) &= \hat{\psi}(\xi)\,\overline{\hat{\psi}(\xi - 2\pi)} + \hat{\psi}(2\xi)\,\overline{\hat{\psi}(2(\xi - 2\pi))} \\ &= b(\xi)b(2\xi)[e^{i\{p(\xi)-p(\xi-2\pi)\}} + e^{i\{p(2\xi)-p(2(\xi-2\pi))\}}], \end{aligned}$$

which is zero almost everywhere because of (v). This finishes the proof of the theorem. □

Theorem 2.1 provides several examples of orthonormal wavelets.

Example 1. Given ε such that $0 < \varepsilon \leq \frac{1}{3}\alpha\,(0 < \alpha \leq \pi)$, we define b on $D_\alpha = [2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha]$ by

$$b(\xi) = \begin{cases} 0, & \text{if } 2\pi - \frac{4}{3}\alpha \leq \xi < 2\pi - \alpha - \varepsilon, \\ \sin\dfrac{\pi}{4}(1 + \dfrac{\xi - 2\pi + \alpha}{\varepsilon}), & \text{if } 2\pi - \alpha - \varepsilon \leq \xi \leq 2\pi - \alpha + \varepsilon, \\ 1, & \text{if } 2\pi - \alpha + \varepsilon < \xi < 2\pi - \frac{2}{3}\alpha. \end{cases}$$

Extend b to S_α using (2.2) and consider that it is zero outside S_α. The function b is continuous and its graph is shown in figure 3.

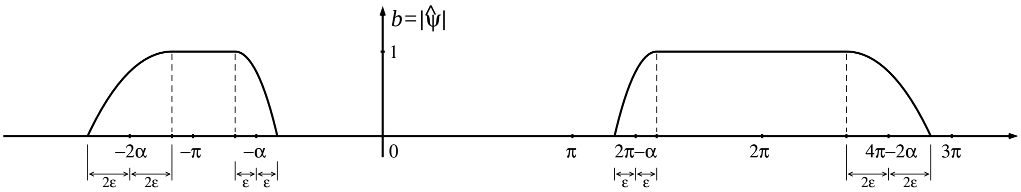

Figure 3

Taking $\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}b(\xi)$, theorem 2.1 tells us that ψ is an orthonormal wavelet. Computer graphs of $\operatorname{Re}\psi$ and $\operatorname{Im}\psi$, for $\alpha = 1$, $\varepsilon = \frac{1}{3}$ are given in figure 4.

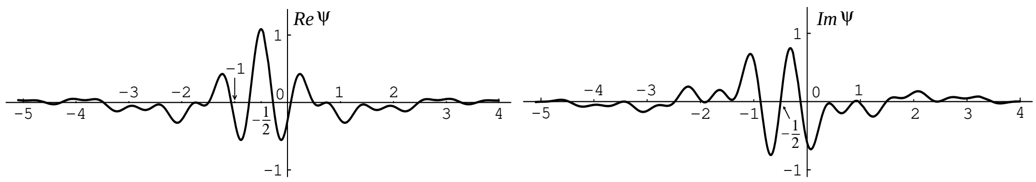

Figure 4

In a similar fashion, examples of wavelets whose Fourier transforms have a greater degree of smoothness can be obtained.

Example 2. Taking $b = \chi_{[2\pi-\frac{4}{3}\alpha, 2\pi-\alpha]}$ on D_α, one of the wavelets produced by theorem 2.1 is given by $\hat{\psi} = \chi_K$, where

$$K = [-\tfrac{8}{3}\alpha, -2\alpha]\cup[-\alpha, -\tfrac{2}{3}\alpha]\cup[2\pi-\tfrac{4}{3}\alpha, 2\pi-\alpha]\cup[2\pi-\tfrac{2}{3}\alpha, 4\pi-\tfrac{8}{3}\alpha]\cup[4\pi-2\alpha, 4\pi-\tfrac{4}{3}\alpha].$$

The graphs of this wavelet for $\alpha = \pi$ and its Fourier transform are given in figure 5.

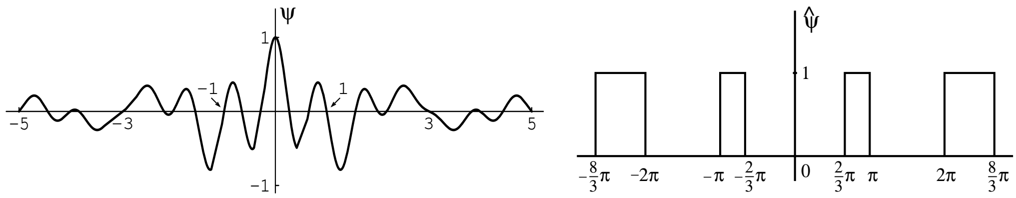

Figure 5

Example 3. Taking $b = \frac{1}{\sqrt{2}}\chi_{D_\alpha}$, the absolute value of the Fourier transform of the wavelets ψ produced by theorem 2.1 is shown in figure 6. When $\alpha = \pi$, $|\hat{\psi}|$ is constant and equal to $\frac{1}{\sqrt{2}}$ on its support.

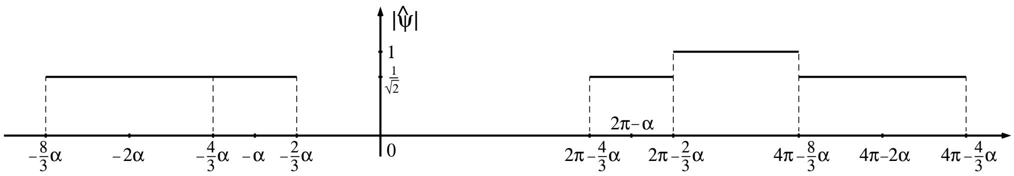

Figure 6

3 MRA and non-MRA wavelets

For the sake of completeness we define a multiresolution analysis (MRA), but assume that the reader is familiar with this concept and how it is used to construct wavelets.

A multiresolution analysis (MRA) consists of a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R})$ satisfying

$$V_j \subset V_{j+1}, \quad j \in \mathbb{Z}, \tag{3.1}$$

$$f \in V_j \text{ if and only if } f(2(\cdot)) \in V_{j+1}, \tag{3.2}$$

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \tag{3.3}$$

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}), \tag{3.4}$$

$$\left.\begin{array}{l}\text{There exists a } \varphi \in V_o, \text{ such that } \{\varphi(\cdot - k) : k \in \mathbb{Z}\} \\ \text{is an orthonormal basis for } V_o.\end{array}\right\} \tag{3.5}$$

The scaling function φ allows us to define the associated low-pass filter, m, by the equality

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi). \tag{3.6}$$

One can show that m is a 2π periodic function satisfying

$$|m(\xi)|^2 + |m(\xi + \pi)|^2 = 1, \qquad \text{for } a.e.\ \xi \in \mathbb{R}. \tag{3.7}$$

A wavelet ψ can then be constructed by the formula

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\,\overline{m(\tfrac{\xi}{2} + \pi)}\,\hat{\varphi}(\tfrac{\xi}{2}). \tag{3.8}$$

It is easily seen that multiplying $\hat{\psi}(\xi)$ by $e^{ip(\xi)}$, with $p(\xi)$ a 2π-periodic function, gives us another wavelet. Condition (v) of theorem 2.1 shows, at least for the wavelets considered there, that this class of factors is natural if we want to define the class of all wavelets associated with this MRA. In general, this periodicity is forced upon us by equation (1.2); a corresponding change of the scaling function will yield these other wavelets so that (3.8) holds for them as well.

If ψ is an orthonormal wavelet we let W_j be the closure of the span of the set $\{\psi_{j,k} : k \in \mathbb{Z}\}$. The W_j's form a mutually orthogonal sequence of closed subspaces of $L^2(\mathbb{R})$. Let V_j be the orthogonal direct sum of the collection $\{W_\ell : -\infty < \ell < j\}$. Then $\{V_j : j \in \mathbb{Z}\}$ satisfy (3.1), (3.2), (3.3) and (3.4). If we can find a function φ such that (3.5) and (3.8) are verified we say that ψ is associated with this MRA or, simply, say that ψ is an MRA-wavelet.

It is natural to inquire whether **all** wavelets can be constructed in this way from an MRA. The example discovered by Journé, presented in the introduction, shows that this is **not** the case in general. It turns out that there is a simple characterization of the MRA-wavelets. This result, involving the function $D_\psi(\xi)$ defined in the proof of proposition 2.3, was obtained independently (and with substantially different proofs) by G. Gripenberg ([Gri]) and one of us (see [Wan]):

Proposition 3.9 *Suppose ψ is an orthonormal wavelet; then it is an MRA-wavelet if and only if $D_\psi(\xi) = 1$ almost everywhere.*

In this section we obtain two results that are connected with the notions just described.

Theorem 3.10 *If ψ is an orthonormal wavelet such that $\hat{\psi}$ is supported in $S_\alpha = [-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha],\ 0 < \alpha \le \pi$ (or $\tilde{S}_\beta = [-4\pi + \frac{4}{3}\beta, \frac{8}{3}\beta],\ 0 < \beta \le \pi$), then ψ is an* MRA-*wavelet.*

Proof : By proposition 3.9 it suffices to show $D_\psi(\xi) = 1$ *a.e.* for the wavelets ψ such that $\operatorname{Supp}\hat{\psi} \subset S_\alpha$. Since D_ψ is 2π-periodic we can restrict our attention to, say, the period interval $[-\frac{2}{3}\alpha, 2\pi - \frac{2}{3}\alpha] = H_\alpha \cup D_\alpha$. By (2.6),

$$D_\psi(\xi) = 1 - |\hat{\psi}(2(\xi - \pi))|^2 = 1 - b^2(2(\xi - \pi)), \quad \text{for } \ a.e.\ \xi \in D_\alpha.$$

But for such ξ, $2(\xi - \pi) \in H_\alpha$. Thus, $D_\psi(\xi) = 1$ *a.e.* on D_α. By (2.5),

$$D_\psi(\xi) = 1 - \sum_{j=-\infty}^{0} |\hat{\psi}(2^j\xi)|^2, \qquad \text{for } \ a.e.\ \xi \in H_\alpha.$$

But $2^j H_\alpha \subseteq H_\alpha$ if $j \le 0$. Thus $D_\psi(\xi) = 1 - 0 = 1$ for almost every $\xi \in H_\alpha$.

□

Next we show that if ψ is a wavelet that is not associated with an MRA, then, in the L^2-norm it is isolated from the MRA-wavelets. More precisely,

Theorem 3.11 *If $\{\psi^n : n = 1, 2, \cdots\}$ is a sequence of* MRA-*wavelets converging in the L^2-norm to a wavelet ψ, then ψ must be an* MRA-*wavelet.*

Remark 6. By theorem 3.10 we see that there are sequences of MRA-wavelets that converge to a function ψ in L^2 ($\|\psi\|_2 = 1$) that is not a wavelet (see Remark 4 in section 2).

Proof : We begin by establishing (2.4) for any $f \in L^2(\mathbb{R})$ when $J = [0, 2\pi)$:

$$\int_0^{2\pi} D_f(\xi)\, d\xi = 2\pi\|f\|_2^2.$$

The following calculation shows this result:

$$\begin{aligned}\int_0^{2\pi} D_f(\xi)\, d\xi &= \int_0^{2\pi} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{f}(2^j(\xi + 2k\pi)|^2 d\xi \\ &= \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} |\hat{f}(2^j\xi)|^2 d\xi = \sum_{j=1}^{\infty} 2^{-j}\|\hat{f}\|_2^2 = 2\pi\|f\|_2^2.\end{aligned}$$

Hence,

$$\int_0^{2\pi} D_{\psi^n - \psi}(\xi)\, d\xi = 2\pi\|\psi^n - \psi\|_2^2$$

tends to zero as n tends to ∞. By Fatou's lemma,

$$\varliminf_{n\to\infty} D_{\psi^n-\psi}(\xi) = 0, \qquad \text{for } \; a.e.\ \xi \in [0, 2\pi].$$

For almost every $\xi \in \mathbb{R}$,

$$\sqrt{D_\psi(\xi)} \leq \sqrt{D_{\psi^n-\psi}(\xi)} + \sqrt{D_{\psi^n}(\xi)} = \sqrt{D_{\psi^n-\psi}(\xi)} + 1.$$

Taking liminf (as $n \to \infty$), we deduce that $\sqrt{D_\psi(\xi)} \leq 1$. But

$$2\pi = 2\pi \|\psi\|_2^2 = \int_0^{2\pi} D_\psi(\xi)\, d\xi \leq \int_0^{2\pi} 1\, d\xi = 2\pi.$$

Hence, $D_\psi(\xi) = 1$ almost everywhere. Proposition 3.9 allows us to finish the proof of the theorem.

□

This theorem is relevant to the problem of smoothing orthonormal wavelets if we combine it with the following result :

Theorem 3.12 ([Aus]) *Suppose that ψ is an orthonormal wavelet that satisfies*

$$\left.\begin{array}{l} |\hat{\psi}| \;\; \textit{is continuous on} \;\; \mathbb{R}, \;\; \textit{and} \\ |\hat{\psi}(\xi)| = O((1+|\xi|)^{-\alpha-\frac{1}{2}}) \qquad \textit{at} \;\; \infty, \;\; \textit{for some} \;\; \alpha > 0. \end{array}\right\} \tag{3.13}$$

Then, ψ is associated with an MRA.

We have given here a formulation of this result which has weaker hypotheses than the original result of P. Auscher. The more general result, proved in [Wan], was obtained after a conversation with P. Auscher.

Combining theorems 3.11 and 3.12 we deduce that it is impossible to obtain a wavelet with continuous Fourier transform if we try to enlarge a "little bit" the support of the Journé wavelet given in the introduction. More generally, we have the following

Corollary 3.14 *If ψ is an orthonormal wavelet which is not associated with an* MRA, *then ψ cannot be approximated in L^2 by orthonormal wavelets that satisfy* (3.13). *In particular, it cannot be approximated by wavelets in the Schwartz class.*

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