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Interpolation of Herz Spaces and Applications

By Eugenio Hernández of Madrid, Dachun Yang of Beijing

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Abstract. C. Herz introduced in [Hr] some new spaces to study properties of functions. An interesting account, with many applications, of some particular cases of the generalized Herz spaces is given in [BS]. In this paper we first identify the duals of the generalized Herz spaces. Then, we characterize their intermediate spaces when the complex method of interpolation for families of spaces is used. Applications are given that show the boundedness of many operators on the generalized Herz spaces.

1. Introduction

We start by giving the necessary definitions. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\widetilde{\chi}_k = \chi_k$ if $k \in \mathbb{N}$ and $\widetilde{\chi}_0 = \chi_{B_0}$, where χ_{C_k} is the characteristic function of C_k.

Definition 1.1. Let $\alpha \in \mathbb{R}, 0 < p \le \infty$ and $0 < q \le \infty$.

(a) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\rm loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \Big\}^{1/p}.$$

(b) The non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\rm loc}(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

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where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\widetilde{\chi}_k\|_{L^q(\mathbb{R}^n)}^p \Big\}^{1/p}.$$

(The usual modifications are made when $p = \infty$ and/or $q = \infty$.)

When $\alpha > 0$, the spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ were introduced by Baernstein and Sawyer in [BS]. The spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are quasi-Banach spaces and if $p, q \geq 1$, $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are Banach spaces.

In §2 of this paper we investigate the dual spaces of the homogeneous and non-homogeneous Herz spaces. We do this in an abstract setting.

Coifman, Cwikel, Rochberg, Sagher and Weiss in [CC1] and [CC2] developed a theory of complex interpolation for families of Banach spaces. Hernández ([He1], [He2]) introduced a theory of complex interpolation for families of Banach lattices and use it to identify the intermediate spaces of many classical spaces, and Tabacco Vignati ([TV1]∼[TV5]) consider the case of quasi-Banach spaces. In §3 of this paper, using the ideas developed by Hernández in [He2], we characterize the intermediate spaces obtained by applying the complex method of interpolation to families of Herz spaces in the case that they are Banach spaces. For the families of Herz spaces which are quasi-Banach spaces, we also characterize their intermediate spaces by complex interpolation by a similar method, using the theory developed in [TV1]∼[TV5]. We do this in §4.

In §5 of this paper, we give some applications of interest. First, using the interpolation theory developed in §3, we show that a wide class of linear operators is bounded on Herz spaces. An interesting fact is that our indices for Herz spaces are critical, which complements the results obtained by Lu and Yang in [LY2]. For sublinear operators, we also obtain similar results, which are very useful in applications to the theory of Herz-type spaces. For example, using these results and the dual theory in §2, we can characterize these Herz spaces of Banach type by means of the generalized Littlewood-Paley $g-$function, which extend similar results for the spaces $L^p(\mathbb{R}^n)$, with $1 < p < \infty$ (see [To]).

2. The dual spaces

In this section, we will investigate the dual spaces for generalized Herz spaces as given in Definition 1.1. We consider this problem in a more abstract setting. Let us first introduce some notation.

Let $\dot{\mathcal{A}} = \{A_j\}_{j=-\infty}^{\infty}$ and $\mathcal{A} = \{A_j\}_{j=0}^{\infty}$, where the A_j's are Banach spaces. For $\alpha \in \mathbb{R}$ and $0 < p \leq \infty$, we define

$$\dot{\ell}_p^{\alpha}(\dot{\mathcal{A}}) = \Big\{ a | a = \{a_j\}_{j=-\infty}^{\infty}, a_j \in A_j \text{ and } \Big(\sum_{j=-\infty}^{\infty} (2^{j\alpha} \|a_j\|_{A_j})^p \Big)^{1/p} < \infty \Big\},$$

and

$$\ell_p^{\alpha}(\mathcal{A}) = \Big\{ a | a = \{a_j\}_{j=0}^{\infty}, a_j \in A_j \text{ and } \Big(\sum_{j=0}^{\infty} (2^{j\alpha} \|a_j\|_{A_j})^p \Big)^{1/p} < \infty \Big\},$$

where the usual modifications are made when $p = \infty$. Moreover,

$$\dot{C}_0^\alpha(\dot{\mathcal{A}}) = \{a | a \in \dot{\ell}_\infty^\alpha(\dot{\mathcal{A}}) \text{ and } 2^{j\alpha}\|a_j\|_{A_j} \to 0 \text{ as } |j| \to \infty\}$$

and

$$C_0^\alpha(\mathcal{A}) = \{a | a \in \ell_\infty^\alpha(\mathcal{A}) \text{ and } 2^{j\alpha}\|a_j\|_{A_j} \to 0 \text{ as } j \to \infty\}.$$

If $\alpha = 0$, these spaces are considered by Triebel in [Tr] (pp 120~127).

Let X^* denote the dual space of X, $\dot{\mathcal{A}}^* = \{A_j^*\}_{j=-\infty}^\infty$ and $\mathcal{A}^* = \{A_j^*\}_{j=0}^\infty$. If $0 < p \leq \infty$ and $1/p + 1/p' = 1 (p' = \infty$, if $0 < p \leq 1)$, then, for $f = \{f_j\}_{j=-\infty}^\infty \in \dot{\ell}_{p'}^{-\alpha}(\dot{\mathcal{A}}^*)$ and $a = \{a_j\}_{j=-\infty}^\infty \in \dot{\ell}_p^\alpha(\dot{\mathcal{A}})$, we define

$$f(a) = \sum_{j=-\infty}^\infty f_j(a_j), \tag{2.1}$$

which is a linear continuous functional over $\dot{\ell}_p^\alpha(\dot{\mathcal{A}})$ and satisfies

$$\|f\|_{(\dot{\ell}_p^\alpha(\dot{\mathcal{A}}))^*} \leq \|f\|_{\dot{\ell}_{p'}^{-\alpha}(\dot{\mathcal{A}}^*)}. \tag{2.2}$$

Similar results hold for the spaces $\ell_p^\alpha(\mathcal{A})$ and $\ell_{p'}^{-\alpha}(\mathcal{A}^*)$.

Theorem 2.1. *Let $\dot{\mathcal{A}}, \mathcal{A}$ be as above, $\alpha \in \mathbb{R}, 0 < p < \infty$ and $1/p + 1/p' = 1$, where $p' = \infty$ if $0 < p \leq 1$. Then $\left(\dot{\ell}_p^\alpha(\dot{\mathcal{A}})\right)^* = \dot{\ell}_{p'}^{-\alpha}(\dot{\mathcal{A}}^*)$, $\left(\dot{C}_0^\alpha(\dot{\mathcal{A}})\right)^* = \dot{\ell}_1^{-\alpha}(\dot{\mathcal{A}}^*)$, $\left(\ell_p^\alpha(\mathcal{A})\right)^* = \ell_{p'}^{-\alpha}(\mathcal{A}^*)$, and $\left(C_0^\alpha(\mathcal{A})\right)^* = \ell_1^{-\alpha}(\mathcal{A}^*)$.*

Proof. We only prove the theorem for the spaces $\dot{\ell}_p^\alpha(\dot{\mathcal{A}})$. We start with the case $1 < p < \infty$. Let $f \in \left(\dot{\ell}_p^\alpha(\dot{\mathcal{A}})\right)^*$. If $a_j \in A_j, j \in \mathbb{Z}$, we define $\delta_j a_j = \{\delta_{jk} a_k\}_{k=-\infty}^\infty$, where

$$\delta_{jk} = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $f_j(a_j) = f(\delta_j a_j)$ is a linear functional over A_j and

$$\sum_{j=-N}^N f_j(a_j) = f\Big(\sum_{j=-N}^N \delta_j a_j\Big).$$

Now, we choose $a'_j \in A_j$ with $\|a'_j\|_{A_j} = 2^{-j\alpha}$ such that $f_j(a'_j)$ is real and

$$f_j(a'_j) \geq 2^{-j\alpha}\|f_j\|_{A_j^*} - \varepsilon_j,$$

where $\varepsilon_j > 0$ are given numbers. We let $a_j = (2^{-j\alpha}\|f_j\|_{A_j^*})^{p'-1} a'_j$. For given $\varepsilon > 0$, we can choose $\varepsilon_j > 0$ such that

$$f_j(a_j) + \varepsilon 2^{-|j|} \geq 2^{-j\alpha p'}\|f_j\|_{A_j^*}^{p'} = 2^{j\alpha p}\|a_j\|_{A_j}^p.$$

Thus,

$$\begin{aligned}\sum_{j=-N}^{N} 2^{-j\alpha p'}\|f_j\|_{A_j^*}^{p'} &\le 4\varepsilon + f\Big(\sum_{j=-N}^{N}\delta_j a_j\Big)\\ &\le 4\varepsilon + \|f\|_{(\dot\ell_p^\alpha(\dot{\mathcal{A}}))^*}\Big(\sum_{j=-N}^{N} 2^{j\alpha p}\|a_j\|_{A_j}^p\Big)^{1/p}.\end{aligned}$$

Letting $\varepsilon\to 0$ and then $N\to\infty$, we obtain

$$\|\{f_j\}_{j=-\infty}^{\infty}\|_{\dot\ell_{p'}^{-\alpha}(\dot{\mathcal{A}}^*)} \le \|f\|_{(\dot\ell_p^\alpha(\dot{\mathcal{A}}))^*}.$$

In addition, (2.1) and (2.2) indicate that we can construct a functional over $\dot\ell_p^\alpha(\dot{\mathcal{A}})$ by using $\{f_j\}_{j=-\infty}^{\infty}\in\dot\ell_{p'}^{-\alpha}(\dot{\mathcal{A}}^*)$. Note that this functional coincides with f for all elements of the form $\sum_{j=-N}^{N}\delta_j a_j$ and these elements are dense in $\dot\ell_p^\alpha(\dot{\mathcal{A}})$; hence, this functional coincides with f on the whole space $\dot\ell_p^\alpha(\dot{\mathcal{A}})$. From this and (2.1)$\sim$(2.2), Theorem 2.1 follows for the case $1<p<\infty$. For $0<p\le 1$ choose $a_j=a_j'$ and $\varepsilon_j=\varepsilon$. For the case $\dot C_0^\alpha(\dot{\mathcal{A}})$ choose $a_j=a_j'$ and $\varepsilon_j=2^{-|j|}\varepsilon$. □

Remark 2.2. The results of Theorem 2.1 for the spaces $\dot\ell_p^0(\mathcal{A})$ and $\dot C_0^0(\mathcal{A})$ with $1\le p<\infty$ and $A_j=A$ for $j\in\mathbb{Z}$ can be found in [Tr] (pp 68).

Now, let $f_k(x)=f(x)\chi_k(x)$ and $\widetilde f_k(x)=f(x)\widetilde\chi_k(x)$, then we have

$$\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}=\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|f_k\|_{L^q(\bar C_k)}\Big\}^{1/p}=\|\{f_k\}_{k=-\infty}^{\infty}\|_{\dot\ell_p^\alpha(\dot{\mathcal{A}})}$$

and

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}=\|\{\widetilde f_k\}_{k=0}^{\infty}\|_{\ell_p^\alpha(\mathcal{A}')},$$

where $\dot{\mathcal{A}}=\{L^q(\bar C_k)\}_{k=-\infty}^{\infty}$, $\mathcal{A}'=\{L^q(\bar B(0,1)),\{L^q(\bar C_k)\}_{k=1}^{\infty}\}$ and $\bar C$ denotes the closure of the set C. From this and using Theorem 2.1, we easily obtain the following corollary. Before we state it, we need more definitions.

Definition 2.3. Let $\alpha\in\mathbb{R}$ and $0<q\le\infty$.

(1) $f\in L^q_{\rm loc}(\mathbb{R}^n\setminus\{0\})$ is said to belong to the space $(\dot K_q^{\alpha,\infty})_0(\mathbb{R}^n)$ if $f\in\dot K_q^{\alpha,\infty}(\mathbb{R}^n)$ and $2^{k\alpha}\|f\chi_k\|_{L^q(\mathbb{R}^n)}\to 0$, as $|k|\to\infty$.

(2) $f\in L^q_{\rm loc}(\mathbb{R}^n)$ is said to belong to the space $(K_q^{\alpha,\infty})_0(\mathbb{R}^n)$ if $f\in K_q^{\alpha,\infty}(\mathbb{R}^n)$ and $2^{k\alpha}\|f\widetilde\chi_k\|_{L^q(\mathbb{R}^n)}\to 0$, as $k\to\infty$.

Remark 2.4. If $1<q<\infty$, the space $(K_q^{-n/q,\infty})_0(\mathbb{R}^n)$ has been introduced by Feichtinger ([Fe]) and García-Cuerva ([Ga]).

Now, we can state our corollary as follows.

Corollary 2.5. *Let* $\alpha \in \mathbb{R}, 0 < p < \infty, 1 \leq q < \infty$, $1/q + 1/q' = 1$ *and* $1/p + 1/p' = 1$, *where* $p' = \infty$ *if* $0 < p \leq 1$. *Then* $\left(\dot{K}_q^{\alpha,p}(\mathbb{R}^n)\right)^* = \dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)$,

$$\left(K_q^{\alpha,p}(\mathbb{R}^n)\right)^* = K_{q'}^{-\alpha,p'}(\mathbb{R}^n),$$

$\left((\dot{K}_q^{-\alpha,\infty})_0(\mathbb{R}^n)\right)^* = \dot{K}_{q'}^{\alpha,1}(\mathbb{R}^n)$ *and* $\left((K_q^{-\alpha,\infty})_0(\mathbb{R}^n)\right)^* = K_{q'}^{\alpha,1}(\mathbb{R}^n)$.

Remark 2.6. For the spaces $K_q^{n(1-1/q),1}(\mathbb{R}^n)$ and $K_q^{-n/q,\infty}(\mathbb{R}^n)$, Corollary 2.5 has been obtained by Feichtinger ([Fe]) and García-Cuerva ([Ga]) (see Theorem 3 in [Fe] and Theorem 1.4 in [Ga]). For the spaces $\dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n)$, and $\dot{K}_q^{-n/q,\infty}(\mathbb{R}^n)$, Corollary 2.5 has been obtained by García-Cuerva and Herrero in [GH] (see Proposition 1.7 in [GH]).

Notice that if $1 \leq p \leq \infty$ and $1 \leq q < \infty$, then the spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are Banach spaces. By the closed-graph theorem, we easily get the following corollary.

Corollary 2.7. *Let* $\alpha \in \mathbb{R}, 1 \leq p, q < \infty$ *and* $1/q + 1/q' = 1 = 1/p + 1/p'$. *Then* $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ *if and only if*

$$\left| \int_{\mathbb{R}^n} f(x)g(x)\,dx \right| < \infty$$

for every $g \in \dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)$ *and, in this case,*

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup\left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)\,dx \right| : \|g\|_{\dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)} \leq 1 \right\}.$$

There exists a similar result for the non-homogeneous space $K_q^{\alpha,p}(\mathbb{R}^n)$.

Remark 2.8. For the space $K_q^{n(1-1/q),1}(\mathbb{R}^n)$, Corollary 2.7 has been obtained by García-Cuerva (see Corollary 1.5 in [Ga]).

3. The complex interpolation for Herz spaces of Banach type

In this section we shall characterize the intermediate spaces obtained by applying the complex method of interpolation to families of Herz spaces which are Banach spaces. We need some notations (see [He1],[He2] for more details).

Let Δ denote the open unit disk in $\mathbb{C}$, the set of complex numbers, and T the boundary of Δ. Let $\{B(\theta)\}_{\theta \in T}$ be a family of Banach spaces. We say that this family is an interpolation family of Banach spaces if each $B(\theta)$ is continuously embedded in a Banach space $(U, \|\cdot\|_U)$, the function $\theta \to \|b\|_{B(\theta)}$ is measurable for each

$$b \in \bigcap_{\theta \in T} B(\theta),$$

and if

$$\beta = \Big\{ b \in \bigcap_{\theta \in T} B(\theta) \;\Big|\; \int_T \log^+ \|b\|_{B(\theta)}\, d\theta < \infty \Big\},$$

we have $\|b\|_U \le c(\theta)\|b\|_{B(\theta)}$ for all $b \in \beta$, where $\log^+ \mathrm{c}(\theta) \in \mathrm{L}^1(\mathrm{T})$. The space β is called the log-intersection space of the given family, $c(\theta)$ is called the log-intersection constant and U is called a containing space. We let $\mathcal{G} = \mathcal{G}(\Delta, B(\cdot))$ be the space of all the $\beta-$valued analytic functions of the form

$$g(z) = \sum_{j=1}^{m} \psi_j(z) b_j$$

for which $\|g\|_\infty = \sup_{\theta \in T} \|g(\theta)\|_{B(\theta)} < \infty$, where $\psi_j \in N^+(\Delta)$, the positive Nevalinna class for Δ (see [Du], chapter 2) and $b_j \in \beta, j = 1, \cdots, m$. (Note: $\mathcal{G} = N^+(B(\cdot))$ in [He2] (pp 246)). The completion of the space $\mathcal{G}$ with respect to $\|\cdot\|_\infty$ is denoted by $\mathcal{F}(B(\cdot))$. The space $[B(\theta)]_z$, which will also be denoted by $B(z)$, consists of all elements of the form $f(z)$ for $f \in \mathcal{F}(B(\cdot))$. A Banach space norm is defined on $B(z)$ by

$$\|v\|_{B(z)} = \inf\{\|f\|_\infty : f \in \mathcal{F}(B(\cdot)), f(z) = v\}$$

for $v \in B(z)$. The space $B(z)$ is called the intermediate space for the family $\{B(\theta)\}_{\theta \in T}$.

We need to define the notion of interpolation of Banach lattices. Let $\{X(\theta)\}_{\theta \in T}$ be a family of Banach lattices on a fixed measure space (M, dx) (see [He2] for the definition of Banach lattice). For $z \in \Delta$, we denote by $[X(\theta)]^z$ the class of measurable functions f on M for which there exist $\lambda > 0$ and a measurable function $F : T \times M \to \mathbb{R}$ with $\|F(\theta, \cdot)\|_{X(\theta)} \le 1$ *a.e.* such that

$$|f(x)| \le \lambda \exp\Big\{ \int_T (\log|\mathrm{F}(\theta, \mathrm{x})|) \mathrm{P_z}(\theta)\, \mathrm{d}\theta \Big\},$$

where P_z is the Poisson kernel for evaluation at z. We let $\|f\|_{[X(\theta)]^z}$ be the infimum of the values of λ that appears in the above inequality.

Now, we can state our theorem as follows:

Theorem 3.1. *Let $\theta \in T$ and $\dot{\mathcal{B}}(\theta) = \{B_k(\theta)\}_{k \in \mathbb{Z}}$. For each $k \in \mathbb{Z}$, we assume that $\{B_k(\theta)\}_{\theta \in T}$ is an interpolation family of Banach spaces with β_k as the log-intersection space and $c_k(\theta)$ as the log-intersection constant. Moreover, we suppose that there exists a measurable function $c(\theta)$ on T such that $c_k(\theta) \le c(\theta)$ uniformly for $k \in \mathbb{Z}$ and $\log^+ \mathrm{c}(\theta) \in \mathrm{L}^1(\mathrm{T})$. If $1 \le p(\theta) \le \infty$, $\alpha(\theta) \in \mathbb{R}$ and there exist constants $\alpha_0, \alpha_1 \in \mathbb{R}$ such that $\alpha_0 \le \alpha(\theta) \le \alpha_1$, then*

$$\Big[\dot{\ell}^{\alpha(\theta)}_{p(\theta)}(\dot{\mathcal{B}}(\theta))\Big]_z = \dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z)),$$

where $z \in \Delta$, $\dot{\mathcal{B}}(z) = \{B_k(z)\}_{k \in \mathbb{Z}}$, $\alpha(z) = \int_T \alpha(\theta) P_z(\theta)\, d\theta$ and $\frac{1}{p(z)} = \int_T \frac{1}{p(\theta)} P_z(\theta)\, d\theta$.

Proof. We check first that $\{\dot{\ell}^{\alpha(\theta)}_{p(\theta)}(\dot{\mathcal{B}}(\theta))\}_{\theta \in T}$ is an interpolation family of Banach spaces. For each $k \in \mathbb{Z}$, let U_k be a containing Banach spaces of the family $\{B_k(\theta)\}_{\theta \in T}$

and $\mathcal{U} = \ell_\infty^{\alpha_0}(\{U_k\}_{k=1}^\infty) + \ell_\infty^{\alpha_1}(\{U_k\}_{k=-\infty}^0)$. We want to prove that $\mathcal{U}$ is a containing space for $\{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))\}_{\theta\in T}$. To see this, let $a \in \bigcap_{\theta\in T}\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))$ and observe that $\|a_k\|_{B_k(\theta)}$ is measurable in θ for every $k\in\mathbb{Z}$. Hence,

$$\|a\|_{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))} = \Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha(\theta)p(\theta)}\|a_k\|_{B_k(\theta)}^{p(\theta)}\Big\}^{1/p(\theta)}$$

is measurable in θ. Moreover, if a belongs to the log-intersection space of the family $\{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))\}_{\theta\in T}$, we have $a_k\in\bigcap_{\theta\in T}B_k(\theta)$ for every $k\in\mathbb{Z}$. Thus,

$$\|a_k\|_{U_k}\le c_k(\theta)\|a_k\|_{B_k(\theta)}$$

and consequently $\{\|a_k\|_{U_k}\}_{k=-\infty}^\infty \in \bigcap_{\theta\in T}\dot\ell_{p(\theta)}^{\alpha(\theta)}$. Since

$$\int_T \log^+\|a\|_{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))}\,d\theta<\infty,$$

we have

$$\int_T \log^+\|\|a_k\|_{U_k}\|_{\dot\ell_{p(\theta)}^{\alpha(\theta)}}\,d\theta \le \int_T\log^+c(\theta)\,d\theta + \int_T\log^+\|a\|_{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))}\,d\theta<\infty.$$

This shows that $\{\|a_k\|_{U_k}\}_{k=-\infty}^\infty\in\Omega$, where Ω denotes the log-intersection space of the family $\{\dot\ell_{p(\theta)}^{\alpha(\theta)}\}_{\theta\in T}$. Since $\{\dot\ell_{p(\theta)}^{\alpha(\theta)}\}_{\theta\in T}$ is an interpolation family, we have

$$\|a\|_{\mathcal{U}}\le\|c_k(\theta)\|a_k\|_{B_k(\theta)}\|_{\dot\ell_{p(\theta)}^{\alpha(\theta)}}\le c(\theta)\|\{a_k\}\|_{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))},$$

where $\int_T\log^+\mathrm{c}(\theta)\,\mathrm{d}\theta<\infty$. This proves that the space $\mathcal{U}$ is the containing space for $\{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))\}_{\theta\in T}$ and that $\{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))\}_{\theta\in T}$ is an interpolation family of Banach spaces.

By an obvious density argument, the inclusion $\Big[\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))\Big]_z\subseteq\dot\ell_{p(z)}^{\alpha(z)}(\dot{\mathcal{B}}(z))$, will follow from the inequality

$$\|\{b_k(z)\}\|_{\dot\ell_{p(z)}^{\alpha(z)}(\dot{\mathcal{B}}(z))}\le\sup_{\theta\in T}\|\{b_k(\theta)\}\|_{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))}, \tag{3.1}$$

which will be shown to be true for any element of the form $b(z)=\{b_k(z)\}_{k\in\mathbb{Z}}$ with $b_k(\xi)=\sum_{j=1}^N\psi_j(\xi)a_k^j$, where $a^j(\theta)=\{a_k^j(\theta)\}_{k=-\infty}^\infty$ belongs to the log-intersection space of the family $\{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))\}_{\theta\in T}$ and $\psi_j(\xi)\in N^+(\Delta)$. To prove (3.1), we observe that for every $k\in\mathbb{Z}, a_k^j\in\bigcap_{\theta\in T}B_k(\theta)$ and consequently $b_k(\xi)\in\mathcal{G}(\Delta,B_k(\cdot))$ for every k. By Theorem (2.1) in [He2] (or see [CC1]), we have

$$\|b_k(z)\|_{B_k(z)}\le\|b\|_\infty\exp\Big\{\int_T P_z(\theta)\log\frac{\|b_k(\theta)\|_{B_k(\theta)}}{\|b\|_\infty}\,d\theta\Big\} \tag{3.2}$$

for every $k\in\mathbb{Z}$, where

$$\|b\|_\infty=\sup_{\theta\in T}\|\{b_k(\theta)\}_{k\in\mathbb{Z}}\|_{\dot\ell_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))}=\sup_{\theta\in T}\Big\{\sum_{k=-\infty}^{\infty}\|b_k(\theta)\|_{B_k(\theta)}^{p(\theta)}2^{k\alpha(\theta)p(\theta)}\Big\}^{1/p(\theta)}.$$

We notice that we can always assume $\|b\|_\infty \neq 0$. Since

$$\left\| \frac{\|b_k(\theta)\|_{B_k(\theta)}}{\|b\|_\infty} \right\|_{\dot{\ell}^{\alpha(\theta)}_{p(\theta)}} \leq 1,$$

the definition of $[\dot{\ell}^{\alpha(\theta)}_{p(\theta)}]^z = \dot{\ell}^{\alpha(z)}_{p(z)}$ (see Corollary (5.4) in [He2]) and (3.2) imply (3.1).

In order to prove the reverse inclusion and the corresponding norm inequality, we need the following lemma, whose proof will be given after the proof of Theorem 3.1. Let $h \in \dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))$, we say h is simple if $h = \{h_k\}_{k \in \mathbb{Z}_1}$, where $\mathbb{Z}_1$ is a finite subset of $\mathbb{Z}$.

Lemma 3.2. *For any given $\varepsilon > 0$, let S_ε be the class of all the simple $h \in \dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))$ such that there exists $H : T \times \mathbb{Z} \to \mathbb{R}$ with $\|H(\theta, \cdot)\|_{\dot{\ell}^{\alpha(\theta)}_{p(\theta)}} \leq 1$ for all $\theta \in T$, satisfying*

$$\|h_k\|_{B_k(\theta)} = (1+\varepsilon)\|h\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \exp\left\{ \int_T P_z(\theta) \log|H(\theta,k)|\, d\theta \right\}$$

and such that the non-zero values of each $|H(\theta,\cdot)|$ have positive upper and lower bounds. Then S_ε is dense in $\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))$.

We proceed now to prove the reverse inclusion in Theorem 3.1. Let $h \in S_\varepsilon$ and write $h = \{h_k\}_{k \in \mathbb{Z}_1}$, where $\mathbb{Z}_1$ is a finite subset of $\mathbb{Z}$. We can find $\psi_j \in \mathcal{F}(B_j(\cdot))$ such that $\psi_j(z) = h_j / \|h_j\|_{B_j(z)}, j \in \mathbb{Z}_1$ and $\|\psi_j\|_\infty \leq 1 + \varepsilon$. Define

$$g_k(\xi) = (1+\varepsilon)\|h\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \exp\left\{ \int_T P_\xi(\theta) \log|H(\theta,k)|\, d\theta \right\} \psi_k(\xi),$$

where $H(\theta,k)$ is the function corresponding to $h \in S_\varepsilon$. Since each ψ_k is a limit of functions in $\mathcal{G}(\Delta, B_k(\cdot))$, one can show that $g = \{g_k\}_{k \in \mathbb{Z}_1} \in \mathcal{F}\Big(\dot{\ell}^{\alpha(\cdot)}_{p(\cdot)}(\dot{\mathcal{B}}(\cdot))\Big)$. An elementary computation shows that $g_k(z) = h_k$; thus $h \in \Big[\dot{\ell}^{\alpha(\theta)}_{p(\theta)}(\dot{\mathcal{B}}(\theta))\Big]_z$. Moreover, since $\|\psi_j(\theta)\|_{B_k(\theta)} \leq \|\psi\|_\infty \leq 1 + \varepsilon$, we deduce

$$\begin{aligned} \|g_k(\theta)\|_{B_k(\theta)} &= (1+\varepsilon)\|h\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} |H(\theta,k)| \|\psi_k\|_{B_k(\theta)} \\ &\leq (1+\varepsilon)^2 \|h\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} |H(\theta,k)|, \end{aligned}$$

which together with $\|H(\theta,k)\|_{\dot{\ell}^{\alpha(\theta)}_{p(\theta)}} \leq 1$ implies

$$\|\{\|g_k(\theta)\|_{B_k(\theta)}\}_{k \in \mathbb{Z}_1}\|_{\dot{\ell}^{\alpha(\theta)}_{p(\theta)}} \leq (1+\varepsilon)^2 \|h\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))}.$$

Hence $h \in \Big[\dot{\ell}^{\alpha(\theta)}_{p(\theta)}(\dot{\mathcal{B}}(\theta))\Big]_z$ and

$$\|h\|_{\left[\dot{\ell}^{\alpha(\theta)}_{p(\theta)}(\dot{\mathcal{B}}(\theta))\right]_z} \leq \|g\|_\infty \leq (1+\varepsilon)^2 \|h\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))}. \tag{3.3}$$

Let now $a \in \dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))$. By Lemma 3.2, we construct a sequence of functions $h_m \in S_\varepsilon$ such that

$$\Big\| a - \sum_{m=1}^{N} h_m \Big\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \le \frac{1}{2^N} \|a\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \tag{3.4}$$

and

$$\|h_m\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \le \frac{1}{2^m}(1+\varepsilon)\|a\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))}, \tag{3.5}$$

$m = 1, 2, \cdots$. By (3.4), the partial sum of the series $\sum_{m=1}^{\infty} h_m$ converges to a in $\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))$. On the other hand, (3.5) and (3.3) imply that $\sum_{m=1}^{\infty} h_m$ also converges in $\Big[\dot{\ell}^{\alpha(\theta)}_{p(\theta)}(\dot{\mathcal{B}}(\theta))\Big]_z$ and its norm is smaller than $(1+\varepsilon)^3\|a\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))}$. But the two series coincide and so we have $a \in \Big[\dot{\ell}^{\alpha(\theta)}_{p(\theta)}(\dot{\mathcal{B}}(\theta))\Big]_z$ with the norm not exceeding

$$(1+\varepsilon)^3\|a\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))}.$$

The result follows from here since ε is arbitrary. This finishes the proof of Theorem 3.1. □

Proof. (Of Lemma 3.2) Let $a \in \dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))$; since $\Big\{\|a_k\|_{B_k(z)}\Big\}_{k\in\mathbb{Z}} \in \dot{\ell}^{\alpha(z)}_{p(z)} = [\dot{\ell}^{\alpha(\theta)}_{p(\theta)}]^z$, there exists $\widetilde{H}(\theta,k)$ with $\|\widetilde{H}(\theta,\cdot)\|_{\dot{\ell}^{\alpha(\theta)}_{p(\theta)}} \le 1$ such that

$$\|a_k\|_{B_k(z)} \le (1+\frac{\varepsilon}{2})\|a\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \exp\Big\{\int_T P_z(\theta)\log|\widetilde{H}(\theta,k)|\,d\theta\Big\}. \tag{3.6}$$

Define $\mathbb{Z}_m = \{k \in \mathbb{Z} : |k| \le m, 1/m \le |\widetilde{H}(\theta,k)| \le m, \text{for each } \theta \in T\}$. Obviously, $\mathbb{Z}_m$ is finite. Let $a^m = \{a_k\}_{k\in\mathbb{Z}_m}$; we have $\|a^m_k - a_k\|_{B_k(z)} \to 0$ as $m \to \infty$ for every k, and $\|a^m_k - a_k\|_{B_k(z)} \le \|a_k\|_{B_k(z)}$. By the Lebesgue's dominated convergence theorem for series, $\|a^m - a\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \to 0$ as $m \to \infty$ and so $\|a^m\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \to \|a\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))}$.

Moreover by (3.6), we obtain

$$\|a^m_k\|_{B_k(z)} \le (1+\frac{\varepsilon}{2})\|a\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \exp\Big\{\int_T P_z(\theta)\log|\widetilde{H}(\theta,k)|\,d\theta\Big\}.$$

Take m so large that $(1+\varepsilon/2)\|a\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \le (1+\varepsilon)\|a^m\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))}$. Then

$$\|a^m_k\|_{B_k(z)} \le (1+\varepsilon)\|a^m\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \exp\Big\{\int_T P_z(\theta)\log|\widetilde{H}(\theta,k)|\,d\theta\Big\}.$$

We have to show that $a^m \in S_\varepsilon$. To do this, we define

$$\alpha^m_k = \frac{\|a^m_k\|_{B_k(z)}}{(1+\varepsilon)\|a^m\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \exp\Big\{\int_T P_z(\theta)\log|\widetilde{H}(\theta,k)|\,d\theta\Big\}},$$

if $k \in \mathbb{Z}_m$ and $\alpha_k^m = 0$ otherwise. Then

$$\begin{aligned} \|a_k^m\|_{B_k(z)} &= (1+\varepsilon)\|a^m\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \exp\Big\{\int_T P_z(\theta) \log|\widetilde{H}(\theta,k)|\,d\theta\Big\}\alpha_k^m \\ &= (1+\varepsilon)\|a^m\|_{\dot{\ell}^{\alpha(z)}_{p(z)}(\dot{\mathcal{B}}(z))} \exp\Big\{\int_T P_z(\theta) \log|\alpha_k^m \widetilde{H}(\theta,k)|\,d\theta\Big\}. \end{aligned}$$

If we define $H(\theta,k) = \alpha_k^m \widetilde{H}(\theta,k)$, then $\|H(\theta,\cdot)\|_{\dot{\ell}^{\alpha(\theta)}_{p(\theta)}} \le \|\widetilde{H}(\theta,\cdot)\|_{\dot{\ell}^{\alpha(\theta)}_{p(\theta)}} \le 1$. Therefore $a^m \in S_\varepsilon$ and we finish the proof of Lemma 3.2. □

For the non-homogeneous case we can prove a similar theorem. In the following, we let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Theorem 3.3. *Let $\theta \in T$ and $\mathcal{B}(\theta) = \{B_k(\theta)\}_{k\in\mathbb{Z}_+}$. For each $k \in \mathbb{Z}_+, \{B_k(\theta)\}_{\theta\in T}$ is an interpolation family of Banach spaces with β_k as the log-intersection space and $c_k(\theta)$ as the log-intersection constant. Moreover, we suppose that there exists a measurable function $c(\theta)$ on T such that $c_k(\theta) \le c(\theta)$ uniformly for $k \in \mathbb{Z}_+$ and $\log^+ c(\theta) \in L^1(T)$. If $1 \le p(\theta) \le \infty$, $\alpha(\theta) \in \mathbb{R}$, $\alpha(\theta) \in L^1(T)$ and there exists a constant $\alpha_0 \in \mathbb{R}$ such that $\alpha_0 \le \alpha(\theta)$, then*

$$\Big[\ell^{\alpha(\theta)}_{p(\theta)}(\mathcal{B}(\theta))\Big]_z = \ell^{\alpha(z)}_{p(z)}(\mathcal{B}(z)),$$

where $z \in \Delta$, $\mathcal{B}(z) = \{B_k(z)\}_{k\in\mathbb{Z}_+}, \alpha(z) = \int_T \alpha(\theta)P_z(\theta)\,d\theta$ and $\frac{1}{p(z)} = \int_T \frac{1}{p(\theta)}P_z(\theta)\,d\theta$.

Let (Y,ν) be a measure space and $1 \le q \le \infty$, then it is easy to prove that

$$L^q(Y) \subset L^1(Y) + L^\infty(Y)$$

and $\|f\|_{L^1(Y)+L^\infty(Y)} \le 2\|f\|_{L^q(Y)}$. From this, we easily obtain

$$K_q^{\alpha,p}(\mathbb{R}^n) \subset \ell_\infty^{\alpha_0}(\{L^1(Y_k) + L^\infty(Y_k)\}_{k\in\mathbb{Z}_+}) \tag{3.7}$$

if $1 \le p,q \le \infty$, $\alpha_0 \le \alpha$, and $Y_k = \bar{C}_k$ if $k \in \mathbb{N}$, $Y_0 = \bar{B}_0$. Also

$$\begin{aligned} \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \subset\ & \ell_\infty^{\alpha_0}(\{L^1(\bar{C}_k) + L^\infty(\bar{C}_k)\}_{k=1}^\infty) \\ &+ \ell_\infty^{\alpha_1}(\{L^1(\bar{C}_k) + L^\infty(\bar{C}_k)\}_{k=-\infty}^0), \end{aligned} \tag{3.8}$$

if $1 \le p,q \le \infty$ and $\alpha_0 \le \alpha \le \alpha_1$.

Using Theorems 3.1 and 3.3 and inclusions (3.7) and (3.8), we immediately obtain the following interpolation theorem for Herz spaces of Banach type.

Theorem 3.4. *Let $\theta \in T, 1 \le p(\theta), q(\theta) \le \infty$, $\frac{1}{p(z)} = \int_T \frac{1}{p(\theta)}P_z(\theta)\,d\theta$ and*

$$\frac{1}{q(z)} = \int_T \frac{1}{q(\theta)} P_z(\theta)\,d\theta,$$

where $z \in \Delta$.

(1) If $\alpha(\theta) \in \mathbb{R}$, $\alpha(\theta) \in L^1(T)$ and there exists $\alpha_0 \in \mathbb{R}$ such that $\alpha(\theta) \ge \alpha_0$, then

$$\Big[K^{\alpha(\theta),p(\theta)}_{q(\theta)}(\mathbb{R}^n)\Big]_z = K^{\alpha(z),p(z)}_{q(z)}(\mathbb{R}^n),$$

where $\alpha(z) = \int_T \alpha(\theta) P_z(\theta)\, d\theta$.

(2) If $\alpha(\theta) \in \mathbb{R}$ *and there exist* $\alpha_0, \alpha_1 \in \mathbb{R}$ *such that* $\alpha_0 \le \alpha(\theta) \le \alpha_1$, *then*

$$\left[\dot{K}_{q(\theta)}^{\alpha(\theta),p(\theta)}(\mathbb{R}^n)\right]_z = \dot{K}_{q(z)}^{\alpha(z),p(z)}(\mathbb{R}^n),$$

where $\alpha(z) = \int_T \alpha(\theta) P_z(\theta)\, d\theta$.

4. The complex interpolation for Herz spaces of quasi-Banach type

In this section we will characterize the intermediate spaces obtained by applying the complex method of interpolation to families of generalized Herz spaces, as given in Definition 1.1, which are quasi-Banach spaces. Let us first introduce some notation (see [TV1]~[TV5] for more details).

For each $\theta \in T$, consider a quasi-Banach space $(B(\theta), \|\cdot\|_{B(\theta)})$, and denote by $c(\theta)$ the constants in the quasi-triangle inequalities. We say that the family $\{B(\theta)\}_{\theta\in T}$ is an interpolation family of quasi-Banach spaces if each $B(\theta)$ is continuously embedded in a Hausdorff topological vector space $\mathcal{U}$, the function $\theta \to \|b\|_{B(\theta)}$ is measurable for each $b \in \bigcap_{\theta\in T} B(\theta)$, and $\log c(\theta) \in L^1(T)$. We define the log-intersection space β of the family $\{B(\theta)\}_{\theta\in T}$, the space $\mathcal{G} = \mathcal{G}(\Delta, B(\cdot))$ of all the $\beta-$valued analytic functions and $\|g\|_\infty$ as we did in the beginning of §3. For every $a \in \beta$ and $z \in \Delta$, we define

$$\|a\|_z = \inf\{\|g\|_\infty : \ g \in \mathcal{G}, g(z) = a\}.$$

If N_z denotes the set of functions of β such that $\|a\|_z = 0$, the completion space $B(z)$ of $(\beta/N_z, \|\cdot\|_z)$ will be called the interpolation space at z of the family $\{B(\theta)\}_{\theta\in T}$. We also denote $B(z)$ by $[B(\theta)]_z$.

On the other hand, let (Y, ν) be a measure space and $0 < q \le \infty$, then it is easy to prove that $L^q(Y) \subset L^0(Y) + L^\infty(Y)$ and

$$\|f\|_{L^0(Y)+L^\infty(Y)} \le 2\|f\|_{L^q(Y)}^{q/(q+1)},$$

where $f \in L^0(Y)$ means that $\|f\|_{L^0(Y)} = |\text{supp f}| < \infty$. Using this inequality and the results in [TV1] (see also [TV5]), we can prove the following complex interpolation theorem for quasi-Banach spaces imitating the proof of Theorem 3.1. We omit the details here.

Theorem 4.1. *For* $\theta \in T$, *let* $\dot{\mathcal{B}}(\theta) = \{B_k(\theta)\}_{k\in\mathbb{Z}}$. *Suppose* $0 < p(\theta) \le \infty, \alpha(\theta) \in \mathbb{R}$ *and* $\alpha(\theta), \frac{1}{p(\theta)} \in L^1(T)$. *Set* $\frac{1}{p(z)} = \int_T \frac{1}{p(\theta)} P_z(\theta)\, d\theta$, *and* $\alpha(z) = \int_T \alpha(\theta) P_z(\theta)\, d\theta$. *Then*

$$\left[\dot{\ell}_{p(\theta)}^{\alpha(\theta)}(\dot{\mathcal{B}}(\theta))\right]_z = \dot{\ell}_{p(z)}^{\alpha(z)}(\dot{\mathcal{B}}(z)),$$

if either of the following conditions holds:

(1) The spaces $B_k(\theta)$ *are Banach spaces for all* $k \in \mathbb{Z}$ *and there are two constants* $\alpha_1, \alpha_0 \in \mathbb{R}$ *such that* $\alpha_0 \le \alpha(\theta) \le \alpha_1$. *In this case,* $\dot{\mathcal{B}}(z) = \{B_k(z)\}_{k\in\mathbb{Z}}$.

(2) $B_k(\theta) = L^{q(\theta)}(Y_k)$, *where* (Y_k, ν_k) *is a measurable space,* $0 < q(\theta) \le \infty$ *and* $\frac{1}{q(\theta)} \in L^1(T)$, *and there are constants* $\alpha_1, \alpha_0 \in \mathbb{R}$ *such that* $\alpha_0 \le \frac{\alpha(\theta)q(\theta)}{q(\theta)+1} \le \alpha_1$. *In this case,* $B_k(z) = L^{q(z)}(Y_k)$, $\frac{1}{q(z)} = \int_T \frac{1}{q(\theta)} P_z(\theta)\, d\theta$ *and* $\dot{\mathcal{B}}(z) = \{L^{q(z)}(Y_k)\}_{k\in\mathbb{Z}}$.

Theorem 4.2. *For* $\theta \in T$, *let* $\mathcal{B}(\theta) = \{B_k(\theta)\}_{k\in\mathbb{Z}_+}$. *Suppose* $0 < p(\theta) \le \infty, \alpha(\theta) \in \mathbb{R}$ *and* $\alpha(\theta), \frac{1}{p(\theta)} \in L^1(T)$. *Set* $\frac{1}{p(z)} = \int_T \frac{1}{p(\theta)} P_z(\theta)\, d\theta$, *and* $\alpha(z) = \int_T \alpha(\theta) P_z(\theta)\, d\theta$. *Then*

$$\left[\ell^{\alpha(\theta)}_{p(\theta)}(\mathcal{B}(\theta))\right]_z = \ell^{\alpha(z)}_{p(z)}(\mathcal{B}(z)),$$

if either of the following conditions holds:

(1) The spaces $B_k(\theta)$ *are Banach spaces for all* $k \in \mathbb{Z}$ *and there exists a constant* $\alpha_0 \in \mathbb{R}$ *such that* $\alpha_0 \le \alpha(\theta)$. *In this case,* $\mathcal{B}(z) = \{B_k(z)\}_{k\in\mathbb{Z}_+}$.

(2) $B_k(\theta) = L^{q(\theta)}(Y_k)$, *where* (Y_k, ν_k) *is a measurable space,* $0 < q(\theta) \le \infty$ *and* $\frac{1}{q(\theta)} \in L^1(T)$, *and there is a constant* $\alpha_0 \in \mathbb{R}$ *such that* $\alpha_0 \le \frac{\alpha(\theta)q(\theta)}{q(\theta)+1}$. *In this case,* $B_k(z) = L^{q(z)}(Y_k)$, $\frac{1}{q(z)} = \int_T \frac{1}{q(\theta)} P_z(\theta)\, d\theta$ *and* $\mathcal{B}(z) = \{L^{q(z)}(Y_k)\}_{k\in\mathbb{Z}_+}$.

As a simple corollary of Theorems 4.1∼4.2, we obtain the results for the complex interpolation for Herz spaces of quasi-Banach type.

Corollary 4.3. *Let* $\theta \in T, \alpha(\theta) \in \mathbb{R}, 0 < p(\theta), q(\theta) \le \infty$, *and* $\alpha(\theta), \frac{1}{p(\theta)}, \frac{1}{q(\theta)} \in L^1(T)$. *Define* $\alpha(z) = \int_T \alpha(\theta) P_z(\theta)\, d\theta$, $\frac{1}{p(z)} = \int_T \frac{1}{p(\theta)} P_z(\theta)\, d\theta$ *and* $\frac{1}{q(z)} = \int_T \frac{1}{q(\theta)} P_z(\theta)\, d\theta$.

(1) If there exist constants $\alpha_1, \alpha_0 \in \mathbb{R}$ *such that* $\alpha_0 \le \frac{\alpha(\theta)q(\theta)}{q(\theta)+1} \le \alpha_1$, *then*

$$\left[\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)}(\mathbb{R}^n)\right]_z = \dot{K}^{\alpha(z),p(z)}_{q(z)}(\mathbb{R}^n).$$

(2) If there is a constant $\alpha_0 \in \mathbb{R}$ *such that* $\alpha_0 \le \frac{\alpha(\theta)q(\theta)}{q(\theta)+1}$, *then*

$$\left[K^{\alpha(\theta),p(\theta)}_{q(\theta)}(\mathbb{R}^n)\right]_z = K^{\alpha(z),p(z)}_{q(z)}(\mathbb{R}^n).$$

5. Some applications

In this section, we shall give some applications. First, we show that a wide class of sublinear operators are bounded on Herz spaces.

Theorem 5.1. *Let* $0 < p \le \infty, 1 \le q \le \infty$, $-n/q < \alpha < n(1 - 1/q)$ *and* $\alpha \ne 0$. *If a sublinear operator* T *satisfies the size conditions*

$$|Tf(x)| \le c\|f\|_{L^1(\mathbb{R}^n)}/|x|^n, \tag{5.1}$$

when supp $f \subset C_k$ *and* $|x| \ge 2^{k+1}$ *with* $k \in \mathbb{Z}$, *and*

$$|Tf(x)| \le c2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}, \tag{5.2}$$

when supp $f \subset C_k$ *and* $|x| \le 2^{k-2}$ *with* $k \in \mathbb{Z}$, *and* T *is bounded on* $L^q(\mathbb{R}^n)$, *then* T *is bounded on* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

Concerning the non-homogeneous spaces, we have a similar theorem as follows.

Theorem 5.2. *Let* p, q *and* α *be as in Theorem 5.1. If a sublinear operator* T *satisfies the size conditions*

$$|Tf(x)| \le c\|f\|_{L^1(\mathbb{R}^n)}/|x|^n, \tag{5.3}$$

when supp $f \subset B(0,1)$ *and* $|x| > 2$ *or* supp $f \subset C_k$ *and* $|x| \ge 2^{k+1}$ *with* $k \in \mathbb{N}$, *and*

$$|Tf(x)| \le c2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}, \tag{5.4}$$

when supp $f \subset C_k$ *and* $|x| \le 2^{k-2}$ *with* $k \ge 2$, *and* T *is bounded on* $L^q(\mathbb{R}^n)$, *then* T *is bounded on* $K_q^{\alpha,p}(\mathbb{R}^n)$.

In what follows, we only prove Theorem 5.1. The proof of Theorem 5.2 is similar.

Proof. (Of Theorem 5.1) If $0 < \alpha < n(1-1/q)$, Theorem 5.1 is just Theorem 2.3 in [LY2], where (5.1) is used. Thus, we only need to prove the case $-n/q < \alpha < 0$. Let $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, $1 \le q \le \infty$ and $0 < p < \infty$; we trivially decompose f into

$$f(x) = \sum_{k=-\infty}^{\infty} \lambda_k b_k(x),$$

$\lambda_k = 2^{k\alpha}\|f\chi_k\|_{L^q(\mathbb{R}^n)}$ and $b_k(x) = f(x)\chi_k(x)/(2^{k\alpha}\|f\chi_k\|_{L^q(\mathbb{R}^n)})$. Then,

$$\begin{aligned}
\|Tf\|^p_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &= \sum_{\ell=-\infty}^{\infty} 2^{\ell\alpha p}\|(Tf)\chi_\ell\|^p_{L^q(\mathbb{R}^n)} \\
&\le c\Big\{\sum_{\ell=-\infty}^{\infty} 2^{\ell\alpha p}\Big(\sum_{k=-\infty}^{\ell+1} \lambda_k\|(Tb_k)\chi_\ell\|_{L^q(\mathbb{R}^n)}\Big)^p\Big\} \\
&\quad + c\Big\{\sum_{\ell=-\infty}^{\infty} 2^{\ell\alpha p}\Big(\sum_{k=\ell+2}^{\infty} \lambda_k\|(Tb_k)\chi_\ell\|_{L^q(\mathbb{R}^n)}\Big)^p\Big\} \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, using the $L^q(\mathbb{R}^n)$-boundedness of T and the fact that α is negative, we obtain

$$\begin{aligned}
I_1 &= c\sum_{\ell=-\infty}^{\infty} 2^{\ell\alpha p}\Big(\sum_{k=-\infty}^{\ell+1} \lambda_k\|(Tb_k)\chi_\ell\|_{L^q(\mathbb{R}^n)}\Big)^p \\
&\le c\sum_{\ell=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\ell+1} \lambda_k 2^{(\ell-k)\alpha}\Big)^p \\
&\le c\begin{cases} \sum_{\ell=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\ell+1} \lambda_k^p 2^{(\ell-k)\alpha p}\Big), 0 < p \le 1 \\ \sum_{\ell=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\ell+1} \lambda_k^p 2^{(\ell-k)\alpha p/2}\Big)\Big(\sum_{k=-\infty}^{\ell+1} 2^{(\ell-k)\alpha p'/2}\Big)^{p/p'}, \\ \qquad\qquad 1 < p < \infty \end{cases}
\end{aligned}$$

$$\leq \quad c \sum_{k=-\infty}^{\infty} \lambda_k^p \leq c\|f\|^p_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},$$

where $1/p + 1/p' = 1$ when $p \in (1, \infty)$. For I_2, note that supp $b_k \subset C_k$. Thus, if $x \in C_\ell$ and $k \geq \ell + 2$, then $|x| \leq 2^\ell \leq 2^{k-2}$. Using (5.2) and Hölder's inequality, we obtain

$$|Tb_k(x)| \leq c2^{-kn}\|b_k\|_{L^1(\mathbb{R}^n)} \leq c2^{-k(\alpha+n/q)}.$$

Thus,

$$\begin{aligned} I_2 \quad &\leq \quad c \sum_{\ell=-\infty}^{\infty} 2^{\ell\alpha p}\Big(\sum_{k=\ell+2}^{\infty} \lambda_k 2^{-k(\alpha+n/q)+\ell n/q}\Big)^p \\ &\leq \quad c \sum_{\ell=-\infty}^{\infty} \Big(\sum_{k=\ell+2}^{\infty} \lambda_k 2^{(\ell-k)(\alpha+n/q)}\Big)^p \\ &\leq \quad c \sum_{k=-\infty}^{\infty} \lambda_k^p \leq c\|f\|^p_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}, \end{aligned}$$

where the last inequality is obtained by the same computation as in the estimate for I_1. The convergence of the series above is obtained by the hypothesis $\alpha + n/q > 0$. This proves the case $0 < p < \infty$. To prove the case $p = \infty$, let $f \in \dot{K}_q^{\alpha,\infty}(\mathbb{R}^n)$ with $1 \leq q < \infty$; then $f(x) = \sum_{k=-\infty}^{\infty} \lambda_k b_k(x)$, λ_k, b_k as above, and $\sup_{k\in\mathbb{Z}} \lambda_k = \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$. Using the above estimates, we obtain that

$$\begin{aligned} \|Tf\|_{\dot{K}_q^{\alpha,\infty}(\mathbb{R}^n)} \quad &= \quad \sup_{\ell\in\mathbb{Z}} 2^{\ell\alpha}\|(Tf)\chi_\ell\|_{L^q(\mathbb{R}^n)} \\ &\leq \quad \sup_{\ell\in\mathbb{Z}} 2^{\ell\alpha}\Big(\sum_{k=-\infty}^{\ell+1} \lambda_k\|(Tb_k)\chi_\ell\|_{L^q(\mathbb{R}^n)}\Big) \\ &\qquad + \sup_{\ell\in\mathbb{Z}} 2^{\ell\alpha}\Big(\sum_{k=\ell+2}^{\infty} \lambda_k\|(Tb_k)\chi_\ell\|_{L^q(\mathbb{R}^n)}\Big) \\ &\leq \quad \sup_{\ell\in\mathbb{Z}}\Big(\sum_{k=-\infty}^{\ell+1} \lambda_k 2^{(\ell-k)\alpha}\Big) + \sup_{\ell\in\mathbb{Z}}\Big(\sum_{k=\ell+2}^{\infty} \lambda_k 2^{(\ell-k)(\alpha+n/q)}\Big) \\ &\leq \quad c\sup_{\ell\in\mathbb{Z}} \lambda_k \leq c\|f\|_{\dot{K}_q^{\alpha,\infty}(\mathbb{R}^n)}. \end{aligned}$$

The proofs of the case where $q = \infty$ are similar. We omit the details. This finishes the proof of Theorem 5.1. □

For linear operators, we can obtain the following stronger conclusions due to the previously proved interpolation theorems.

Corollary 5.3. *Let $0 < p \leq \infty, 1 \leq q \leq \infty$ and $-n/q < \alpha < n(1 - 1/q)$. If T is linear and respectively satisfies the conditions in Theorem 5.1 and Theorem 5.2, then the results in Theorem 5.1 and Theorem 5.2 are also true for $\alpha = 0$.*

Proof. We only prove the homogeneous case. The proof of the non-homogeneous case is similar. Let T be as in the corollary, $0 < p \le \infty$ and $1 \le q \le \infty$. By Theorem 5.1, we obtain

$$\|Tf\|_{\dot K_q^{\alpha_i,p}(\mathbb{R}^n)} \le c\|f\|_{\dot K_q^{\alpha_i,p}(\mathbb{R}^n)},$$

where $i = 0,\ 1$, $-n/q < \alpha_0 < 0$ and $0 < \alpha_1 < n(1-1/q)$. Let $\theta = \alpha_0/(\alpha_0 - \alpha_1)$, then $\theta \in (0,1)$ and $(1-\theta)\alpha_0 + \theta\alpha_1 = 0$. By Corollary 4.3, we know that

$$\left[\dot K_q^{\alpha_0,p}(\mathbb{R}^n), \dot K_q^{\alpha_1,p}(\mathbb{R}^n)\right]_\theta = \dot K_q^{0,p}(\mathbb{R}^n).$$

From this and Theorem 3.1 in [TV2] (pp 10), we deduce that

$$\|Tf\|_{\dot K_q^{0,p}(\mathbb{R}^n)} \le c\|f\|_{\dot K_q^{0,p}(\mathbb{R}^n)}.$$

This finishes the proof of Corollary 5.3. □

Conditions (5.1)∼(5.4) can be replaced by the following stronger condition but more appealing. To be precise, we have the following simple corollary.

Corollary 5.4. *Let p, q and α be as in Theorem 5.1. If a sublinear operator T satisfies the following condition*

$$|Tf(x)| \le c\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy, \quad x \notin \text{supp } f, \tag{5.5}$$

for any integrable function f compactly supported and T is bounded on $L^q(\mathbb{R}^n)$, then T is bounded on $\dot K_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$. Moreover, if T is linear, then T is also bounded on $\dot K_q^{0,p}(\mathbb{R}^n)$ and $K_q^{0,p}(\mathbb{R}^n)$.

We should point out that such condition was first introduced by Soria and Weiss in [SW]. As it was pointed out in [SW] and [LY2], (5.5) is satisfied by many operators in harmonic analysis, such as Calderón-Zygmund operators , the Carleson maximal operator, C. Fefferman's singular multiplier operator, R. Fefferman's singular integral operator, F. Ricci-E. M. Stein's oscillatory integral operator and the Bochner-Riesz means at the critical index. If $0 < \alpha < n(1-1/q)$, Theorems 5.1 and 5.2 are just Theorem 2.3 in [LY2], which plays a very important role in [LY1] and [LY3]. It is worth pointing out that the indices $-n/q$ and $n(1-1/q)$ in Theorems 5.1 ∼5.2 and Corollaries 5.3∼5.4 are critical. In other words, when $\alpha \ge n(1-1/q)$ or $\alpha \le -n/q$, these theorems and corollaries are false. In [LY2], Lu and Yang have worked the case $\alpha \ge n(1-1/q)$. For the case $\alpha \le -n/q$, we let $n = 1$, $T = H$, the Hilbert transform, and show that Corollary 5.4 is not true. Define $f_k(x) = \chi_{\{2^{k-1}<x\le 2^k\}}(x)$ with $k \ge 2$. When $0 < x \le 2^{k-2}$, then

$$|Hf_k(x)| = \left|p.v.\frac{1}{\pi}\int_{2^{k-1}}^{2^k} \frac{1}{x-y}\,dy\right| \ge c,$$

and c is independent of k. Thus, if Corollary 5.4 with $0 < p < \infty$ were true, then

$$c2^{k(\alpha+n/q)} \ge \|Hf_k\|_{\dot K_q^{\alpha,p}(\mathbb{R})} \ge c_1\Big\{\sum_{\ell=-\infty}^{k-2} 2^{\ell(\alpha+n/q)p}\Big\}^{1/p} = \infty$$

if $\alpha \le -n/q$, which proves the result for the homogeneous spaces. For the non-homogeneous case, we have

$$(5.6)\qquad \begin{aligned} c2^{k(\alpha+n/q)} &\ge \|Hf\|_{K_q^{\alpha,p}(\mathbb{R}^n)} \ge c_1\Big\{\sum_{\ell=0}^{k-2} 2^{\ell(\alpha+n/q)p}\Big\}^{1/p} \\ &= c_1\Big\{\frac{1-2^{(k-1)(\alpha+n/q)p}}{1-2^{(\alpha+n/q)p}}\Big\}^{1/p}. \end{aligned}$$

In (5.6), if $\alpha < -n/q$, then $c_1\{1-2^{(\alpha+n/q)p}\}^{-1/p} \le 0$ and if $\alpha = -n/q$, then $c \ge \infty$. This is impossible. Thus Corollary 5.4 and therefore Corollary 5.3 and Theorems 5.1∼5.2 are false for the Hilbert transform with $0 < p < \infty$, $1 < q < \infty$, $\alpha \le -n/q$, or $\alpha \ge n(1-1/q)$. We point out that for $p = \infty$, $\alpha = -n/q$ and $1 < q < \infty$, Theorems 5.1∼5.2 are true for the Hardy-Littlewood maximal function $M(f)(x)$ (see [Ga]).

In the following, using Theorems 5.1∼5.2, we will characterize these Herz spaces of Banach type by means of the generalized Littlewood-Paley $g-$function.

Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be radial, supp $\psi \subset \{x \in \mathbb{R}^n : |x| \le 1\}$ and $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$. For $t > 0$, let $\psi_t(x) = t^{-n}\psi(x/t)$ and we define

$$g(f)(x) = \Big\{\int_0^\infty |f * \psi_t(x)|^2 \,\frac{dt}{t}\Big\}^{1/2}.$$

It is easy to verify that $g(f)$ satisfies the conditions (5.1)∼(5.4). Thus, we obtain the following results.

Theorem 5.5. *Let $0 < p \le \infty, 1 < q < \infty, -n/q < \alpha < n(1-1/q)$ and $\alpha \ne 0$. If $g(f)$ is as above, then $\|g(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le c\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}$ and*

$$\|g(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)} \le c\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}.$$

Moreover, if also $1 \le p < \infty$, then, $\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le c\|g(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}$ and

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} \le c\|g(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)}.$$

Proof. We only need to show the second part of Theorem 5.5, since the first part follows from Theorems 5.1∼5.2. We only do it for homogeneous spaces. For non-homogeneous spaces, the proof is similar. To do this, let $h \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\|h\|_{\dot K_{q'}^{-\alpha,p'}(\mathbb{R}^n)} \le 1$, where $1/p + 1/p' = 1 = 1/q + 1/q'$. Then, by Corollary 2.5, we obtain

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} f(x)h(x)\,dx\Big| &\le c\int_{\mathbb{R}^n}\int_0^\infty |f*\psi_t(x)||h*\psi_t(x)|\,\frac{dt}{t}\,dx \\ &\le c\int_{\mathbb{R}^n} g(f)(x)g(h)(x)\,dx \\ &\le c\|g(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}\|g(h)\|_{\dot K_{q'}^{-\alpha,p'}(\mathbb{R}^n)}, \end{aligned}$$

where in the first inequality, we have used the fact $\int_0^\infty \frac{|\widehat{\psi}(t)|^2}{t}\,dt = c < \infty$ (see [To], pp 313). Since $-n/q < \alpha < n(1-1/q)$, then $-n/q' < -\alpha < n(1-1/q')$. By the first part of this theorem, we obtain that

$$\|g(h)\|_{\dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)} \leq c\|h\|_{\dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)} \leq c.$$

Thus,

$$\Big|\int_{\mathbb{R}^n} f(x)h(x)\,dx\Big| \leq c\|g(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

The desired result follows from Corollary 2.7. This finishes the proof of Theorem 5.5. □

Corollary 5.6. *Let $g(f)$ be as above. If $1 < p, q < \infty$ and $\alpha = n(1/p - 1/q)$, then*

$$c_1\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leq \|g(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leq c_2\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

There is a similar theorem for the non-homogeneous Herz spaces.

Proof. For $p \neq q$, Corollary 5.6 follows from Theorem 5.5. For $p = q$, and hence $\alpha = 0$, the Corollary is just (3.8) and (3.10) in [To] (pp 312∼313). □

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Eugenio Hernández
Matemáticas
Universidad Autónoma de Madrid
28049 Madrid
Spain
(e-mail: eugenio.hernandez@uam.es)

Dachun Yang
Department of Mathematics
Beijing Normal University
100875 Beijing
The People's Republic of China
(e-mail: dcyang@bnu.edu.cn)