UNIVERSIDADE DE CABO VERDE
PROGRAMA REGIONAL DE DOUTORAMIENTO EM MATEMÁTICA
BASES IN BANACH SPACES - 2024 EXERCISES. DEADLINE: MAY 15, 2024

1.A. Let $(X, \| \, \|)$ be a normed vector space. Show the following:
(i) $|\|x\| - \|y\|| \leq \|x - y\|$ (ii) $\| \, \| : X \mapsto \mathbb{R}$ is continuous.

1.B. Show that if $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a normed vector space $(X, \| \, \|)$, there exists a subsequence $\{v_{n_k}\}_{n=1}^{\infty}$ such that $\|v_{n_{k+1}} - v_{n_k}\| \leq \frac{1}{2^k}$, $k = 1, 2, 3, \ldots$

2.A. Observe that if $a, b \geq 0$, then $\sqrt{ab} \leq \frac{a+b}{2}$.
(a) Show that if $x_i, y_i \in \mathbb{R}, i = 1, \ldots, n$, then the following Cauchy-Schwarz inequality holds:

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}$$

(Hint: Let $A = \sum_{i=1}^{n} |x_i|^2$ and $B = \sum_{i=1}^{n} |y_i|^2$. Use the observation with $\frac{|x_i|^2}{A}$ and $\frac{|y_i|^2}{A}$.)
(b) If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ show that

$$\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

is a norm.

2.B. Show that if X is a closed subspace or a Banach space $(\mathbb{B}, \| \, \|)$, then $(\mathbb{B}, \| \, \|)$ is also a Banach space.

2.C. Prove that $(L^{\infty}(X), \| \, \|_{\infty})$ is a Banach space.

2.D. Let $1 \leq p < q < \infty$. Show that if $f(x) = x^{-1/q}\chi_{[0,1]}(x)$, then $f \in L^p(\mathbb{R}, dx)$, but $f \notin L^q(\mathbb{R}, dx)$.

2.E. Let $1 \leq p < q < \infty$. Show that if $g(x) = x^{-1/p}\chi_{[1,\infty}(x)$, then $g \in L^q(\mathbb{R}, dx)$, but $g \notin L^p(\mathbb{R}, dx)$.

2.F. Let $1 \leq p < q < \infty$ and $x = (x_i)_{i=1}^{\infty}$.
(1) If $\|x\|_p = 1$, show that $\|x\|_q \leq 1$.
(2) Show that if $x \in \ell^p$, then $\|x\|_q \leq \|x\|_p$ (This shows that $\ell^p \subset \ell^q$.)

3.A. Let $(X_1, \| \, \|_1)$ and $(X_2, \| \, \|_2)$ be two normed vector spaces. Let $\mathcal{B}(X_1, X_2)$ be the set of all linear bounded operators from X_1 into X_2. The set $\mathcal{B}(X_1, X_2)$ is a vector spaces with the operations $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ and $(\alpha T)(x) = \alpha(T(x))$. For $T \in \mathcal{B}(X_1, X_2)$ define

$$\|T\|_{op} := \sup_{x \neq 0, x \in X_1} \frac{\|T(x)\|_2}{\|x\|_2}.$$

Show that $\| \, \|_{op}$ is a norm in the vector space $\mathcal{B}(X_1, X_2)$.

4.A. Show that in a pre-Hilbert space, the map $(x,y) \mapsto \langle x,y\rangle$ is continuous, that is if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

4.B. Show that $C([0,1])$ is not a Hilbert space with the inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx\,.$$

Hint: Consider the sequence

$$f_n(x) = \sqrt{n(x-1/2)}\chi_{(\frac{1}{2},\frac{1}{2}+\frac{1}{n}]}(x) + \chi_{(\frac{1}{2}+\frac{1}{n},1]}(x).$$

4.C. Let $(\mathbb{H}, \langle\ ,\ \rangle)$ be a Hilbert space. Let $\mathcal{B} = \{e_n\}_{n=1}^{\infty}$ be an orthonormal system in $\mathbb{H}$. Show that the following two conditions are equivalent:

(i) $\mathcal{B}$ is a basis for $\mathbb{H}$.

(ii) (Parseval's identity) For all $x, y \in \mathbb{H}$, $\langle x,y\rangle = \sum_{n=1}^{\infty} \langle x, e_n\rangle \overline{\langle y, e_n\rangle}$.

(Hint: Use Theorem 4.7 from class notes.)

4.D. Show that in $L^2([a,b])$ the set

$$\{e_n(x) = \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} x} : n \in \mathbb{Z}\}, \qquad (T = b-a),$$

is an orthornormal system with the inner product given by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx\,.$$

5.A. Prove that if $\{y_n\}_{n=1}^{\infty}$ satisfies $\lim_{nto\infty} \|y_n - y\| = 0$, then $\lim_{n\to\infty} \|y_n\| = \|y\|$.

5.B. Assume that $\mathcal{B} = \{x_n\}_{n=1}^{\infty}$ is a basis for a Banach space $(\mathbb{B}, \|\ \|)$. Show that for every sequence of scalars $\{\alpha_n\}_{n=1}^{\infty}$ and all positive integers m, n such that $m > n$, it holds

$$\left\| \sum_{k=1}^{m} \alpha_n x_n \right\| \leq K_b \left\| \sum_{k=1}^{n} \alpha_n x_n \right\|,$$

where K_b denotes the basis constant given in definition 5.5

5.C. (**Sum basis in** c_0) Let $\mathcal{C} = \{\delta_n\}_{n=1}^{\infty}$ be the canonical basis of $(c_0, \|\ \|_\infty)$. Define

$$f_n := \delta_1 + \delta_2 + \cdots + \delta_n, \quad n \in \mathbb{N}.$$

Show that $\mathcal{S} := \{f_n\}_{n=1}^{\infty}$ is a basis for $(c_0, \|\ \|_\infty)$. Find its basis constant.

(Hint: Show that for $z = \{z_n\}_{n=1}^{\infty} \in c_0$, $z = \sum_{n=1}^{\infty} (z_n - z_{n+1}) f_n$ with convergence in $\|\ \|_\infty$.)

5.D. (**Difference basis in** ℓ^1) Let $\mathcal{C} = \{\delta_n\}_{n=1}^{\infty}$ be the canonical basis of ℓ^1. Define

$$x_1 := \delta_1 \qquad x_n = \delta_n - \delta_{n-1}, \quad n = 2, 3, \dots$$

Show that $\mathcal{D} := \{x_n\}_{n=1}^{\infty}$ is a monotone basis of $(\ell^1, \| \, \|_1)$.
(Hint: Start showing that for $\{b_n\}_{n=1}^{M}$ scalars, then

$$\left\| \sum_{n=1}^{M} b_n x_n \right\|_1 = \sum_{n=1}^{M} |b_n - b_{n+1}| + |b_M| \, ,$$

and use Proposition 5.9.)

6.A. Assume that $\mathcal{B} = \{x_n\}_{n=1}^{\infty}$ is a basis for a Banach space $(\mathbb{B}, \| \, \|)$. Show that the following are equivalent:

1. $\mathcal{B}$ is unconditional
2. There exists $K, 1 \leq K < \infty$, such that for all $N \in \mathbb{N}$

$$\left\| \sum_{k=1}^{N} \epsilon_n a_n x_n \right\| \leq K \left\| \sum_{k=1}^{N} a_n x_n \right\| \, ,$$

for all scalars $a_1, \dots, a_N$ and all $\epsilon_n = \pm 1$ for all $n = 1, 2, \dots, N$.

3. There exists $K, 1 \leq K < \infty$, such that for all $N \in \mathbb{N}$

$$\left\| \sum_{k=1}^{N} \beta_n a_n x_n \right\| \leq K \left\| \sum_{k=1}^{N} a_n x_n \right\| \, ,$$

for all scalars $a_1, \dots, a_N$ and all $\beta_n = 0$ or 1 for all $n = 1, 2, \dots, N$.

6.B. Show that $\mathcal{C} = \{\delta_n\}_{n=1}^{\infty}$ is an unconditional basis of $(c_0, \| \, \|_\infty)$ and find is unconditional constant $K_u(\mathcal{C})$.

6.C. Show that an orthonormal basis $\mathcal{B} = \{e_n\}_{n=1}^{\infty}$ in a Hilbert spaces $(\mathbb{H}, \langle \, , \rangle))$ is unconditional and find its unconditional constant $K_u(\mathcal{B})$.

7.A. Define $S_N f(x) = \displaystyle\sum_{k=-N}^{N} \widehat{f}(k) e^{2\pi i k x}$, $x \in \mathbb{R}$. Prove that $S_N f(x) = \displaystyle\int_0^1 f(t) D_N(x-t) dt$, where

$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi i k t} = \left\{ \begin{array}{cc} 2N+1 & \text{if } t = 0 \\ \frac{\sin(2N+1)\pi t}{\sin \pi t} & \text{if } t \neq 0 \end{array} \right\},$$

is called the Dirichlet kernel.

7.B. Recall that the Fejér kernel is defined as $F_N(t) = \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} D_k(t)$, where $D_k(t)$ is the Dirichlet kernel of problem **7.A.** Show that the Fejér kernel is given by

$$F_N(t) = \left\{ \begin{array}{cc} \frac{1}{N} (\frac{\sin \pi N t}{\sin \pi t})^2 & \text{if } t \in \mathbb{R} \setminus \mathbb{Z} \\ N & \text{if } t \in \mathbb{Z} \end{array} \right\},$$

7.C. Show that for every $0 < \delta < \frac{1}{2}$, $\lim_{N\to\infty} \int_{\delta<|t|\le\frac{1}{2}} |F_N(t)|dt = 0$, where $F_N(t)$ is the Dirichlet kernel of exercise **7.B.**

7.D. Show that the Fourier series does not converge for all functions in $(C(\mathbb{T}), \| \; \|_\infty)$, that is, show that there exists $f \in C(\mathbb{T})$ such that $\lim_{N\to\infty} \|S_N f - f\|_\infty \neq 0$.

8.A. Show that for $1 \le p < \infty$, and $n = 2^j + k$, $\|h_n\|_p = 2^{-j/p}$, where h_n denotes the Haar function.

8.B. Write $\chi_{[0,1/4)}$ and $\chi_{[1/2,3/4)}$ as linear combination of elements of the Haar system.

8.C. Show that if $x, y \in \mathbb{R}, x \ge 0, y \ge 0$, and for all $1 \le p < \infty$,

$$(x+y)^p \le 2^{p-1}(x^p + y^p).$$

(Hint: Show that $\varphi(x) = \dfrac{2^{p-1}(x^p+1)}{(x+1)^p} \ge 1, x \ge 0$.)

8.D. Show that for $p = 2$, the set $\mathcal{H}^{(2)} = \{h_n^{(2)}\}_{n=1}^\infty$ is and orthonormal system of $L^2([0,1])$.

8.E. For $n = 2^j + k, j = 0, 1, 2, \ldots, k = 1, 2, \ldots, 2^j$, define $\varphi_n(x) = 2^{j+1} \int_0^x h_n(t)dt$, where h_n denote the Haar functions. Show that

$$\varphi_n(x) = \left\{ \begin{array}{ll} 2^{j+1}x - (2k-2) & \text{if } \frac{2k-2}{2^{j+1}} \le x \le \frac{2k-1}{2^{j+1}} \\ -2^{j+1}x + 2k & \text{if } \frac{2k-1}{2^{j+1}} \le x \le \frac{2k}{2^{j+1}} \\ 0 & \text{otherwise} \end{array} \right\},$$

9.A. For $\psi \in L^2(\mathbb{R})$ define $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), j, k \in \mathbb{Z}$. Show that for all $j, k \in \mathbb{Z}$, $\|\psi_{j,k}\|_2 = \|\psi\|_2$.

9.B. Show that if $g : \mathbb{R} \to \mathbb{C}$ has a radial decreasing L^1-majorant (RDM), then $g \in L^p(\mathbb{R})$, $1 \le p \le \infty$.

9.C. Let $\beta = \{\beta_{j,k}\}_{j,k\in\mathbb{Z}}$ where $\beta_{j,k} = 1$ for a finite number of indices and $\beta_{j,k} = 0$ for the rest. For $f \in L^p(\mathbb{R})$, define

$$T_\beta f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \beta_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k},$$

where the functions $\psi_{j,k}$ are defined in **9.A.** Writing $\langle f, \psi_{j,k} \rangle$ as an integral and interchanging the sum and the integral, show that

$$(T_\beta f)(x) = \int_{\mathbb{R}} K_\beta(x,y) f(y) dy \,,$$

where

$$K_\beta(x,y) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \beta_{j,k} \psi_{j,k}(x) \overline{\psi_{j,k}(y)} \,.$$