11. WAVELET BASIS IN $L^p(\mathbb{R})$, $1<p<\infty$

We know, by Def 10.2, that if $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet, the set

$$W(\psi) := \{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\} \qquad (11.1)$$

is an orthonormal basis of $L^2(\mathbb{R})$. Therefore, by Exercise 7.D $W(\psi)$ is an unconditional basis for the Hilbert space $L^2(\mathbb{R})$. We also have the expansion

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}, \qquad f \in L^2(\mathbb{R}) \qquad (11.2).$$

In this section we will show that expansions such as (11.2) also hold in $L^p(\mathbb{R})$, $1<p<\infty$, and that $W(\psi)$ is an unconditional basis for these spaces.

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1. When $f \in L^p(\mathbb{R})$, the convergence of expansions such as (11.2) is in the sense of the "rectangular" partial sums:

$$S_{m,n}(f) := \sum_{|j|\le m} \sum_{|k|\le n} \langle f, \psi_{j,k}\rangle \psi_{j,k}, \quad m,n\in\mathbb{Z} \qquad (11.3)$$

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2. We need to make sure that ψ and $W(\psi)$ belong to $L^p(\mathbb{R})$. We will always assume that ψ has a radial decreasing L^1-majorant W (see Def 10.5 and Exercise 10.E). In this case, ψ and $W(\psi)$ belong to $L^p(\mathbb{R})$, $1\le p\le\infty$, (see Exercise 10.E.

We will use item (iv) of Theorem 7.8 to show that $W(\psi)$ is an unconditional basis of $L^p(\mathbb{R})$, $1<p<\infty$.

Let $\beta = \{\beta_{j,k} : j, k \in \mathbb{Z}\}$ be a sequence such that $\beta_{j,k} = 1$ for a finite number of indices and $\beta_{j,k} = 0$ for the remaining indices. Define the linear operator

$$T_\beta f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \beta_{j,k} \langle f, \psi_{j,k}\rangle \psi_{j,k}, \qquad (11.4)$$

for $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$.

Exercise 11.A (~~Assume that ψ has a radial decreasing~~ L^1-~~majorant W)~~ Writting $\langle f, \psi_{j,k}\rangle$ and interchanging the summation and the integral show that

$$(T_\beta f)(x) = \int_{\mathbb{R}} k_\beta(x,y) f(y)\, dy$$

where

$$k_\beta(x,y) = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \beta_{j,k} \psi_{j,k}(x) \overline{\psi_{j,k}(y)}$$

S/ $$T_\beta f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \beta_{j,k} \left(\int_{\mathbb{R}} f(y) \overline{\psi_{j,k}(y)}\, dy \right) \psi_{j,k}(x)$$

$$= \int_{\mathbb{R}} \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \beta_{j,k} \psi_{j,k}(x) \overline{\psi_{j,k}(y)} \right) f(y)\, dy$$

$$= \int_{\mathbb{R}} k_\beta(x,y) f(y)\, dy$$

where the sums and the integral can be interchanged because we have a finite number of summands, due to the definition of the sequence β.

Lemma 11.1 Assume that ψ has a radial decreasing L^1-majorant. Then, $k_\beta(x,y)$, as in Exercise 11.A satisfies

$$|k_\beta(x,y)| \leq C \sum_{j\in\mathbb{Z}} 2^j W\left(2^j \frac{|x-y|}{2}\right),$$

where C depends only on W.

Proof. It follows from Lemma 10.6:

$$|k_s(x,y)| \leq \sum_{j\in\mathbb{Z}} 2^j \sum_{k\in\mathbb{Z}} |\psi(2^j x-k)||\psi(2^j y-k)|$$

$$\leq \sum_{j\in\mathbb{Z}} 2^j \sum_{k\in\mathbb{Z}} W(|2^j x-k|)\, W(|2^j y-k|)$$

$$\leq C \sum_{j\in\mathbb{Z}} 2^j W\left(2^j \frac{|x-y|}{2}\right).$$

QED.

We shall use Lemma 11.1 to show that T_s is a Calderón-Zygmund operator.

Def 11.2. A Calderón-Zygmund operator T is a bounded linear operator on $L^2(\mathbb{R})$ such that, for $x \notin \operatorname{supp}(f)$

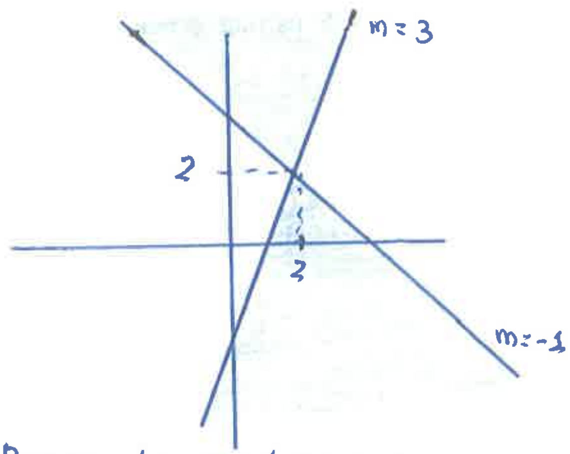
$$(Tf)(x) = \int_{\mathbb{R}} k(x,y) f(y)\,dy$$

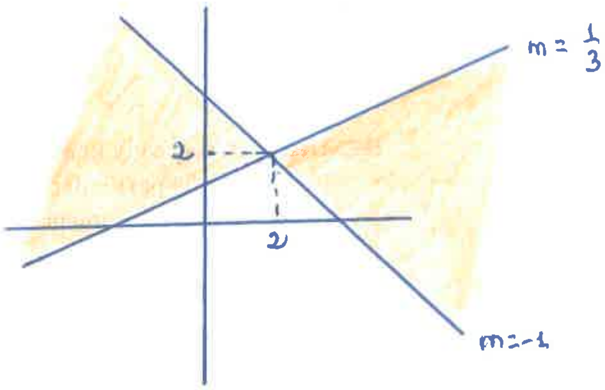
where the kernel k is a measurable function satisfying

$$|k(x,y)| \leq \frac{C_1}{|x-y|} \qquad (11.5)$$

$$|k(x_0,y) - k(x,y)| \leq \frac{C_2|x-x_0|}{|x-y|^2} \quad \text{if } |x-x_0| \leq \frac{1}{2}|x-y| \qquad (11.6)$$

$$|k(x,y_0) - k(x,y)| \leq \frac{C_3|y-y_0|}{|x-y|^2} \quad \text{if } |y-y_0| \leq \frac{1}{2}|x-y| \qquad (11.7)$$





Region $|x-2| \leq \frac{1}{2}|x-y|$

Remark. The Calderón - Zygmund operator with kernel

$$K(x,y)=\frac{1}{\pi}\,\frac{1}{x-y}$$

is called the Hilbert transform H. It is easy to see that K satisfies (11.5), (11.6) and (11.7). The reason why the Hilbert transform H is bounded in $L^2(\mathbb{R})$ is because

$$(Hf)^{\wedge}(\omega)=-i(\operatorname{sign}\omega)\,\hat{f}(\omega).$$

Theorem 11.3 Suppose that T is a Calderón - Zygmund operator as given in Definition 11.2.

Then, T extends to a bounded operator in $L^p(\mathbb{R})$, $1<p<\infty$, with operator norm depending only on $\|T\|_{L^2(\mathbb{R})}$ and the constants that appear in inequalities (11.5), (11.6) and (11.7)

The proof of this result can be found in J. Duoandikoetxea's book (Theorem 5.10). .). It will be proved during the course on Harmonic Analysis that will be given by J. Conde in May in the University of Cabo Verde.

We return to the main purpose of this setion, to prove that most wavelet systems are unconditional bases of $L^p(\mathbb{R})$, $1<p<\infty$.

~~We are going to assume that ψ is a wavelet such that~~

$$\cancel{\hat{\psi}(0)=\int_{\mathbb{R}}\psi(x)\,dx=0}$$

~~This is not a big restriction since as soon as $\hat{\psi}$ be continuous at zero, $\hat{\psi}(0)=0$ (See Proposition 2.1 in Section 7.2 of Hernández-Weiss).~~

Theorem 11.4 Suppose that $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet such that ψ and ψ' have a common radial decreasing L^1-majorant W satisfying

$$\int_0^\infty s W(s)\,ds = C' < \infty$$

Then, the operators T_β defined in (11.4) are bounded on $L^p(\mathbb{R})$, $1<p<\infty$, with norm bounded by a constant independent of the sequences β considered in (11.4).

P/ The operators T_β are bounded on $L^2(\mathbb{R})$ since $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$; by Plancherel

$$\|T_\beta f\|_2^2 = \sum_{j,k\in\mathbb{Z}}\sum |\beta_{jk} \langle f, \psi_{j,k}\rangle|^2 \leq \sum_{j,k\in\mathbb{Z}}\sum |\langle f, \psi_{j,k}\rangle|^2 = \|f\|_2^2$$

They all have norm 1.

It remains to show (11.5), (11.6) and (11.7) with constants independent of $\beta = (\beta_n)_{n=1}^\infty$ in order to apply Thm 11.3 to conclude the proof of Thm 11.4.

To prove (11.5) we use lemma 11.1 to obtain

$$|K_\beta(x,y)| \leq C \sum_{j\in\mathbb{Z}} 2^j W\left(2^j \frac{|x-y|}{2}\right)$$

$$= C \sum_{j=-\infty}^{0} 2^j W\left(2^j \frac{|x-y|}{2}\right) + C \sum_{j=1}^{\infty} 2^j W\left(2^j \frac{|x-y|}{2}\right)$$

$$\leq 2C \int_0^1 W\left(t \frac{|x-y|}{2}\right) dt + 2C \int_1^\infty W\left(t \frac{|x-y|}{2}\right) dt$$

$$= 2c\int_0^{\infty} \frac{2}{|x-y|}\, W(s)\,ds = \frac{4c}{|x-y|}\,\|W\|_{L^1([0,\infty))}.$$

To prove (11.6) we assume $x_0 < x$. We shall show that

$$\left|\frac{\partial k_\beta}{\partial x}(x,y)\right| \le \frac{c}{|x-y|^2} \qquad (11.8)$$

From (11.8), we apply the Mean Value Theorem to obtain a point $x' \in (x_0, x)$ such that

$$|k_\beta(x_0,y) - k_\beta(x,y)| \le |x_0 - x|\left|\frac{\partial k_\beta(x',y)}{\partial x}\right| \le \frac{c|x-x_0|}{|x'-y|^2}$$

Observe that $|x-x_0| \le \frac{1}{2}|x-y|$ implies $y \notin (x_0, x)$. If $y \ge x$ we have $|x'-y| \ge |x-y| \ge \frac{1}{2}|x-y|$. If $y \le x_0$,

$$|x'-y| \ge |x_0-y| \ge |x-y| - |x-x_0| \ge \frac{1}{2}|x-y|$$

Hence,

$$|k_\beta(x_0,y) - k_\beta(x,y)| \le \frac{4c|x-x_0|}{|x-y|} \quad \text{if} \quad |x-x_0| \le \frac{1}{2}|x-y|.$$

We need to show (11.8). We use lemma 10.6 and that W is decreasing to obtain

$$\left|\frac{\partial k_\beta}{\partial x}(x,y)\right| = \left|\sum_{j,k\in\mathbb{Z}}\sum \beta_{jk}\, 2^j\, 2^j\, \psi'(2^jx-k)\,\overline{\psi(2^jy-k)}\right|$$

$$\le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{2j}\, W(|2^jx-k|)\, W(|2^jy-k|) \overset{\text{Lemma 10.6}}{\le}$$

$$\le c\sum_{j\in\mathbb{Z}} 2^{2j}\, W\!\left(2^j\frac{|x-y|}{2}\right) \le 2c\int_0^{\infty} t\,W\!\left(t\,\frac{|x-y|}{2}\right)dt$$

$$= 2c\int_0^{\infty} \frac{2s}{|x-y|}\, W(s)\, \frac{2\,ds}{|x-y|} = \frac{8c}{|x-y|^2}\int_0^{\infty} s\,W(s)\,ds = \frac{8cc'}{|x-y|^2}$$

This proves (11.8) and consequently (11.6). The proof of (11.7) is similar.

QED

Theorem 11.5 Suppose that $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet such that ψ and ψ' have a common radial decreasing L^1-majorant W satisfying

$$\int_0^\infty s\,W(s)\,ds = C_1 < \infty.$$

Then, the system $\underbrace{\{\psi_{j,k} : j,k \in \mathbb{Z}\}}_{W(\psi)=}$ is an unconditional basis for $L^p(\mathbb{R})$, $1<p<\infty$.

Proof We start by showing that $W(\psi)$ is a basis for $L^p(\mathbb{R})$, $1<p<\infty$. Let $S_{m,n}$ be the "retangular" partial sums of the wavelet expansion of f; that is

$$S_{m,n}f = \sum_{|j|\le m} \sum_{|k|\le n} \langle f, \psi_{j,k}\rangle \psi_{j,k}, \quad f \in L^p(\mathbb{R})$$

We want to show that given $f \in L^p(\mathbb{R})$, $1<p<\infty$, and $\varepsilon>0$, there exists $N = N(\varepsilon, f) \in \mathbb{N}$ such that for all $m,n \ge N$,

$$\|f - S_{m,n}f\|_p < \varepsilon. \tag{11.9}$$

Let $C = \sup \|T_\beta\|_{op}$, where the supremum is taken over all sequences $\beta = \{\beta_{j,k}\}_{j,k\in\mathbb{Z}}$ as in (11.4). Since $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, there exists $g \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ such that

$$\|f-g\|_p \le \frac{\varepsilon}{C+3} \tag{11.10}$$

We have

$$\|f - S_{m,n}f\|_p \le \|f-g\|_p + \|g - S_{m,n}g\|_p + \|S_{m,n}(g-f)\|_p \tag{11.11}$$

By Theorem 11.4

$$\|S_{m,n}(g-f)\|_p \le C\|g-f\|_p \le \frac{C\varepsilon}{C+3}, \tag{11.12}$$

It remains to estimate $\|g - S_{m,n}g\|_p$ for $g \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$. By duality, and the density of $L^2(\mathbb{R}) \cap L^{p'}(\mathbb{R})$ in $L^{p'}(\mathbb{R})$, with $\frac{1}{p}+\frac{1}{p'}=1$, we can find $h \in L^2(\mathbb{R}) \cap L^{p'}(\mathbb{R})$ such that

$$\|g - S_{m,n}g\|_p \leq \left| \int_{\mathbb{R}} [g(x) - S_{m,n}g(x)]\,\overline{h(x)}\,dx \right| + \frac{\varepsilon}{C+3}$$

By the Cauchy-Schwarz inequality

$$\left| \int_{\mathbb{R}} [g(x) - S_{m,n}g(x)]\,\overline{h(x)}\,dx \right| \leq \|g - S_{m,n}g\|_2\, \|h\|_2$$

Since $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, there exists $N \in \mathbb{N}$ such that

$$\|g - S_{m,n}g\|_2 \leq \frac{\varepsilon}{\|h\|_2 (C+3)}$$

Therefore

$$\|g - S_{m,n}g\|_p \leq \|h\|_2 \frac{\varepsilon}{\|h\|_2 (C+3)} + \frac{\varepsilon}{C+3} \qquad (11.13)$$

Replace (11.10), (10.12) and (11.13) in (11.11) to obtain

$$\|f - S_{m,n}f\|_p \leq \frac{\varepsilon}{C+3} + \frac{2\varepsilon}{C+3} + \frac{C\varepsilon}{C+3} = \varepsilon.$$

This proves (11.9).

The uniqueness of the representation

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k} \qquad (11.14)$$

with convergence in $L^p(\mathbb{R})$, $1<p<\infty$, follows from the orthonormality of $W(\psi)$: if (11.14) holds,

$$\langle f, \psi_{m,n}\rangle = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\langle \psi_{j,k}, \psi_{m,n}\rangle = c_{m,n}.$$

The unconditionality of $\mathcal{W}(\psi)$ follows from Theorem 11.4 and item (iv) of Theorem 7.8. QED

Remarks Theorem 11.5 cannot be apply to the Haar wavelet $h = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, since h is not derivable. Nevertheless, for the case of $\psi = h$, the operator T_β given in (11.4) can be prove to satisfy a weak type (1.1) estimate; that is

$$|\{x \in \mathbb{R} : |T_\beta f(x)| > \alpha\}| \leq \frac{C}{\alpha} \|f\|_1 \qquad (11.15)$$

for all $f \in L^1(\mathbb{R})$ and $\alpha > 0$. Using the boundedness of T_β on $L^2(\mathbb{R})$ and the Marcinkiewicz interpolation theorem, it can be proved that $\mathcal{H} = \{h_{j,k} : j, k \in \mathbb{Z}\}$ is an unconditional basis for $L^p(\mathbb{R})$, $1 < p < \infty$.

The proof of (11.15) for a Calderón-Zygmund operator T will be given by J. Conde in the course on Harmonic Analysis in May at the University of Cabo Verde.
