# 10. Wavelet Expansions in $L^p(\mathbb{R})$, $1 \leq p < \infty$

**Def 10.1.** Let $f: \mathbb{R} \to \mathbb{C}$. For $y_0 \in \mathbb{R}$ the translation operator is $(T_{y_0} f)(x) = f(x - y_0)$. The dilation operator for $a > 0$ is $(D_a f)(x) = a^{1/2} f(ax)$

Recall the definition of the Fourier transform: for $f \in L^1(\mathbb{R})$,

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \omega} dx$$

The above integral is finite since $|\hat{f}(\omega)| \leq \int_{\mathbb{R}} |f(x)| dx = \|f\|_1 < \infty$. The definition of Fourier transform can be define, by density of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, to be an isometry in $L^2(\mathbb{R})$. Therefore, we have,

$$\|\hat{f}\|_2^2 = \|f\|_2^2 \qquad , f \in L^2(\mathbb{R}) \qquad (10.1)$$

which is called Plancherel's identity.

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**Def 10.2** A function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet for $L^2(\mathbb{R})$ if the set

$$\{\psi_{j,k}(x) := D_{2^j} T_k \psi(x) = 2^{j/2} T_k \psi(2^j x) = 2^{j/2} \psi(2^j x - k) : j, k \in \mathbb{Z}\}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

Let $h(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$. The collection $\{h_{j,k} : j, k \in \mathbb{Z}\} = \mathcal{H}^{(2)}$ is called the Haar system in $\mathbb{R}$. It can be proved (see Exercise 9.C for the Haar system in $L^2([0,1))$) that $\mathcal{H}^{(2)}$

is an orthonormal system in $L^2(\mathbb{R})$. Moreover, it is also a basis for $L^2(\mathbb{R})$.

Exercise 10.A Inspired by the proof of Proposition 9.2, show that the Haar system $\{h_{j,k} : j,k \in \mathbb{Z}\} = \mathcal{H}^2$ is a basis for $L^p(\mathbb{R})$, $1 \leq p < \infty$

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Exercise 10.B Show

(a) $\|\psi_{j,k}\|_2 = \|\psi\|_2 \quad \forall j,k \in \mathbb{Z}$

(b) $(\psi_{j,k})^\wedge(\omega) = 2^{-j/2} e^{-2\pi i k \frac{\omega}{2^j}} \hat{\psi}(\omega/2^j)$ , $\forall j,k \in \mathbb{Z}$

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Not many orthonormal wavelets were known before the 1980's. It was Y. Meyer, with his student P.G. Lemarie, who discovered that there exists many wavelets which are band-limited (that is, their Fourier transform has compact support). Latter, around 1987, S. Mallat and Y. Meyer developed the concept of Multiresolution Analysis (MRA) to generate wavelets. This allowed I. Daubechies to developed compactly supported wavelets, essencial in the compression algorithms of modern signal processing techniques for compression.

Def 10.3 (Multiresolution Analysis, MRA).

An MRA is a set $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ such that

(1) $V_j \subset V_{j+1} \quad \forall j \in \mathbb{Z}$ (2) $f \in V_j \iff D_2 f \in V_{j+1} \quad \forall j \in \mathbb{Z}$

(3) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ (4) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$

(5) There exists $\varphi \in V_0$ such that $\{T_k \varphi : k \in \mathbb{Z}\}$ is an orth. basis of $V_0$.

The function $\varphi$ is called the scaling function of the MRA

Example 10.4 Let $\varphi = \chi_{[0,1]}$ and $V_0 = \overline{\operatorname{span}\{T_k \varphi : k \in \mathbb{Z}\}}$ in $L^2(\mathbb{R})$. Then $\{T_k \varphi : k \in \mathbb{Z}\}$ is an orthonormal basis of $V_0$. Define, for $j \in \mathbb{Z}$,

$$V_j = \overline{\operatorname{span}\{D_{2^j} T_k \varphi : k \in \mathbb{Z}\}}$$

Since $\operatorname{supp} D_{2^j} T_k \varphi = \operatorname{supp} \varphi_{j,k} = I_{j,k} = [\frac{k}{2^j}, \frac{k+1}{2^j})$, the functions in $V_j$ are functions in $L^2(\mathbb{R})$ that are constant on the disjoint interval $\{I_{j,k} : k \in \mathbb{Z}\}$. From here, it can be seen that $\{V_j : j \in \mathbb{Z}\}$ is an MRA.

Def 10.3. Let $(\{V_j : j \in \mathbb{Z}\}, \varphi)$ be an MRA. A low-pass filter is a periodic function $h \in L^2_{per}([0,1])$ such that

$$\hat{\varphi}(2\omega) = h(\omega)\hat{\varphi}(\omega)$$

Exercise 10.C Find the low-pass filter of the MRA given in Example 10.4

S/ $$\hat{\varphi}(\omega) = \int_0^1 e^{-2\pi i x\omega} dx = \left[\frac{e^{-2\pi i x \omega}}{-2\pi i \omega}\right]_0^1 = \frac{e^{-2\pi i\omega} - 1}{-2\pi i \omega}$$

$$= \frac{1 - e^{-2\pi i \omega}}{2\pi i \omega}$$

$$\therefore \hat{\varphi}(2\omega) = \frac{1 - e^{-4\pi i \omega}}{4\pi i\omega} = \frac{(1 + e^{-2\pi i\omega})(1 - e^{-2\pi i \omega})}{4\pi i \omega} =$$

$$= \frac{1}{2}(1 + e^{-2\pi i\omega})\hat{\varphi}(\omega) \Rightarrow h(\omega) = \frac{1}{2}(1 + e^{-2\pi i \omega})$$

Since $V_j \subset V_{j+1}$, we can define $W_j$ to be the orthogonal complement of $V_j$ in $V_{j+1}$, that is

$$V_j \oplus W_j = V_{j+1} \;, \quad j \in \mathbb{Z} \qquad (10.2)$$

By item (3) of MRA

$$\bigoplus_{j=-\infty}^{J-1} W_j = V_J \;, \quad J \in \mathbb{Z} \qquad (10.3)$$

and by item (4) of MRA

$$\bigoplus_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R}) \qquad (10.4)$$

**Theorem 10.4** (S. Mallat, 1989)

Let $\{V_j : j \in \mathbb{Z}\}$ be an MRA with scaling function $\varphi$. Let $h$ be its low-pass filter. Define $\psi$ by the equation

$$\hat{\psi}(\omega) = \nu(\omega)\, e^{-\pi i \omega}\, \overline{h\left(\frac{\omega}{2} + \frac{1}{2}\right)}\, \hat{\varphi}\left(\frac{\omega}{2}\right) \qquad (10.5)$$

where $\nu(\omega)$ is 1-periodic and $|\nu(\omega)| = 1$ a.e. $\omega \in \mathbb{R}$. The function $\psi$ belongs to $W_0$. Moreover, for all $j \in \mathbb{Z}$ $\{\psi_{j,k} : k \in \mathbb{Z}\}$ is an orthonormal basis of $W_j$. By (10.4) $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal wavelet.

**Exercise 10.D** Show that for the MRA of Example 10.4, the wavelet $\psi$ given in (10.5) with $\nu(\omega) = -1$, is the Haar wavelet $h(x) = \chi_{[0,1/2)} - \chi_{[1/2,1)}$. (Hint: compute $\hat{\psi}$ with (10.5) and $\hat{h}$ to obtain the same result)

S/ Using Exercise 10.6

$$\hat{\psi}(\omega) = -e^{-\pi i \omega}\, \overline{\frac{1}{2}\left(1 + e^{-2\pi i\left(\frac{\omega}{2} + \frac{1}{2}\right)}\right)}\; \frac{e^{-\pi i \omega} - 1}{-\pi i \omega} =$$

$$= \frac{1}{2} e^{-\pi i \omega} \left(1 + e^{\pi i \omega}\right) \frac{e^{-\pi i \omega} - 1}{\pi i \omega}$$

$$= \frac{\left(e^{-\pi i \omega} - 1\right)^2}{2\pi i \omega}$$

On the other hand

$$\hat{h}(\omega) = \int_0^{1/2} e^{-2\pi i \omega x} dx - \int_{1/2}^{1} e^{-2\pi i \omega x} dx$$

$$= \left[\frac{e^{-2\pi i \omega x}}{-2\pi i \omega}\right]_0^{1/2} - \left[\frac{e^{-2\pi i \omega x}}{-2\pi i \omega}\right]_{1/2}^{1}$$

$$= \frac{e^{-\pi i \omega} - 1}{-2\pi i \omega} - \frac{e^{-2\pi i \omega} - e^{-\pi i \omega}}{-2\pi i \omega}$$

$$= \frac{e^{-2\pi i \omega} - 2e^{-\pi i \omega} + 1}{2\pi i \omega} = \frac{\left(e^{-\pi i \omega} - 1\right)^2}{2\pi i \omega}$$

Comparing both equalities we obtain $\psi = h$

There are many other wavelets. In [HW] you can find the Shannon wavelet, the Franklin wavelet (related to the Schauder system of Exercise 9.D, the spline wavelets, the Lemarié-Meyer wavelets, and the most popular Daubechies wavelets (Chapters 2 and 3 of [HW]).

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If $\psi$ is an orthonormal wavelet in $L^2(\mathbb{R})$, the set

$$W := \{\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$$

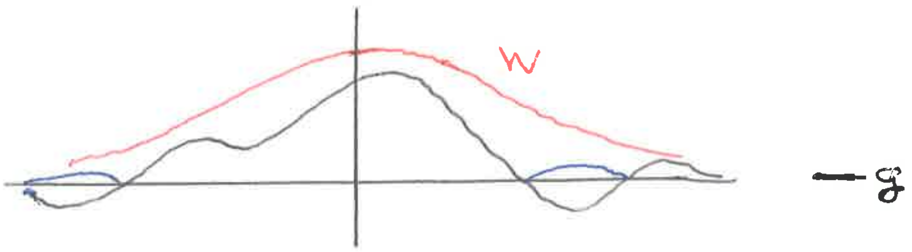
is an orthonormal basis of $L^2(\mathbb{R})$. By Example 6.9 it is a Schauder basis of $L^2(\mathbb{R})$ and by Exercise 7.D it is unconditional. In this case, any $f \in L^2(\mathbb{R})$ can be written as

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k}\rangle \psi_{j,k}. \qquad (10.5)$$

We will show that expansions as (10.5) are also valid in $L^p(\mathbb{R})$, $1\leq p<\infty$. From now on all wavelets considered here will be arising from an MRA. We need to make sure that the set $W$ is contained in $L^p(\mathbb{R})$, $1\leq p<\infty$.

**Def 10.5** Given $g:\mathbb{R}\to\mathbb{C}$, we say that a function $W:[0,\infty)\to\mathbb{R}^+$ is a radial decreasing $L^1$-majorant of $g$ if $|g(x)|\leq W(|x|)$ and

(a) $W\in L^1([0,\infty))$, (b) $W$ is non-increasing (c) $W(0)<\infty$




**Exercise 10.E** Show that if $g:\mathbb{R}\to\mathbb{C}$ has a radial decreasing $L^1$-majorant, then $g\in L^p(\mathbb{R})$, $1\leq p\leq\infty$.

S/ Since $W$ is bounded by $W(0)$ due to (b) and (c), then $\|g\|_\infty\leq W(0)$. For $1\leq p<\infty$, choose $x_0\in\mathbb{R}^+$ such that $W(x_0)\leq 1$ (this can be done because $W\in L^1([0,\infty))$. Since $W(x)\leq 1$ for all $x\geq x_0$ by (b), we have

$$\|g\|^p_{L^p(\mathbb{R})}\leq 2\left\{\int_0^{x_0}|W(x)|^p dx+\int_{x_0}^\infty |W(x)|^p dx\right\}$$

$$\leq 2\left\{W(0)\cdot x_0+\int_{x_0}^\infty |W(x)|\,dx\right\}$$

$$\leq 2\left\{W(0)\cdot x_0+\|W\|_{L^1([0,\infty))}\right\}<\infty.$$

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Notice also that $g_{j,k}\in L^p(\mathbb{R})$, because

$$\|g_{j,k}\|_p^p=\int_{\mathbb{R}}2^{jp/2}|g(2^jx-k)|^p dx=\int_{\mathbb{R}}2^{jp/2}|g(y)|^p\frac{dy}{2^j}=2^{j\left(\frac{p}{2}-1\right)}\|g\|_p^p<\infty$$

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Let $\psi$ be a wavelet that arises from an MRA with scaling function $\varphi$. For the increasing sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ consider the orthogonal projections $P_{V_j}$ from $L^2(\mathbb{R})$ onto $V_j$ given by

$$P_{V_j}(f) = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{j,k}\rangle \varphi_{j,k} \quad , \quad f \in L^2(\mathbb{R}). \qquad (10.6)$$

Also consider the orthogonal projections $P_{W_j}$ from $L^2(\mathbb{R})$ onto $W_j$ given by

$$P_{W_j}(f) = \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k} \quad , \quad f \in L^2(\mathbb{R}) \qquad (10.7)$$

Since $V_j \oplus W_j = V_{j+1}$ it follows that $P_{V_j} + P_{W_j} = P_{V_{j+1}}$. We also consider the partial sum operator given by

$$S^{\sigma}_{j,k} f := \sum_{\ell=-\infty}^{j-1} \sum_{m\in\mathbb{Z}} \langle f, \psi_{\ell,m}\rangle \psi_{\ell,m} + \sum_{m=1}^{k} \langle f, \psi_{j,\sigma(m)}\rangle \psi_{j,\sigma(m)}$$

where $f \in L^2(\mathbb{R})$, $j, k \in \mathbb{Z}$ and $\sigma$ is any permutation of $\mathbb{Z}$.

Since $V_j = \bigoplus_{\ell=-\infty}^{j-1} W_j$ we have

$$S^{\sigma}_{j,k} f = P_{V_j} f + \sum_{m=1}^{k} \langle f, \psi_{j,\sigma(m)}\rangle \psi_{j,\sigma(m)} \qquad (10.8)$$

These operators are defined on $L^2(\mathbb{R})$. We will study if this operators could be defined in $L^p(\mathbb{R})$ and study their convergence properties as $j \to \infty$.

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**Lemma 10.6** Let $W$ be a function with properties (a), (b) and (c) of Def 10.5. Then

$$\sum_{k\in\mathbb{Z}} W(|x-k|)\, W(|y-k|) \leq C\, W\!\left(\frac{|x-y|}{2}\right) \quad \forall x, y \in \mathbb{R}$$

where $C$ is a constant which depends only on $W$.

P/ Since $|x-y| \le |x-k| + |y-k|$, either $|x-k| \ge \frac{1}{2}|x-y|$ or $|y-k| \ge \frac{1}{2}|x-y|$. Then, by the properties of $W$

$$\sum_{k\in\mathbb{Z}} W(|x-k|)W(|y-k|) \le W\left(\tfrac{1}{2}|x-y|\right) \sum_{k:\,|x-k|\ge\frac{1}{2}|x-y|} W(|y-k|)$$

$$+ W\left(\tfrac{1}{2}|x-y|\right) \sum_{k:\,|x-k|<\frac{1}{2}|x-y|} W(|x-k|) \le W\left(\tfrac{1}{2}|x-y|\right)\left\{\sum_{k\in\mathbb{Z}} W(|y-k|) + \sum_{k\in\mathbb{Z}} W(|x-k|)\right\}$$

$$\le C\|W\|_1 W\left(\tfrac{1}{2}|x-y|\right).$$

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Suppose that the scaling function $\varphi$ has a radial decreasing $L^1$-majorant $W$. Then if $f\in L^p(\mathbb{R})$, $1\le p\le\infty$

$$(P_{V_j}f)(x) = \sum_{k=-\infty}^{\infty}\left(\int_{\mathbb{R}} f(y)\overline{\varphi_{j,k}(y)}\,dy\right)\varphi_{j,k}(x)$$

Since $\sum_{k=-\infty}^{\infty} |f(y)\varphi_{j,k}(y)||\varphi_{j,k}(x)| \le |f(y)| \sum_{k=-\infty}^{\infty} 2^j|\varphi(2^jy-k)||\varphi(2^jx-k)|$

$\le 2^j|f(y)|CW\left(\frac{2^j}{2}|x-y|\right)$ and

$$\int_{\mathbb{R}} 2^j|f(y)|W\left(\frac{2^j}{2}|x-y|\right)dy \le 2^jC\|f\|_p \left\|\int_0^{\infty} W\left(\frac{2^j}{2}|x-y|\right)^q dy\right. < \infty$$

by Lemma 10.6 and also because $W\in L^q([0,\infty))$ for all $1\le p\le\infty$.

By the Lebesgue dominated convergence theorem, we can interchange the sum and the integral to obtain

$$(P_{V_j}f)(x) = \int_{\mathbb{R}} f(y)\sum_{k=-\infty}^{\infty} 2^j\varphi(2^jx-k)\overline{\varphi(2^jy-k)}$$

$$=: \int_{\mathbb{R}} 2^j K_\varphi(2^jx, 2^jy)f(y)\,dy \qquad (10.9)$$

where $$K_\varphi(x,y) = \sum_{k=-\infty}^{\infty} \varphi(x-k)\overline{\varphi(y-k)} \qquad (10.10)$$

If we assume also that the wavelet $\psi$ has a radial decreasing $L^1$-majorant, similar expressions to (10.9) and (10.10) can be obtained for the orthogonal projection $P_{W_j} f$ and the partial sums $S^{\sigma}_{J,k} f$.

**Proposition 10.7** Suppose the scaling function $\varphi$ has a radial decreasing $L^1$-majorent $W$. Then, there exists $C>0$, independent of $j$'s such that for all $f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, $1 \le p \le \infty$, we have

$$\|P_{V_j} f\|_p \le C \|W\|_{L^1([0,\infty))} \|f\|_p$$

**Proof.** If $p=\infty$, by (10.9), (10.10) and lemma 10.6,

$$|P_{V_j} f(x)| \le C \int_{\mathbb{R}} 2^j W\left(2^j \frac{|x-y|}{2}\right) |f(y)|\, dy \le C \|f\|_\infty \cdot 2 \int_0^\infty W(|z|)\, 2\, dz$$

$$= 4C \|f\|_\infty \|W\|_{L^1([0,\infty))}.$$

For $p=1$, again by (10.9), (10.10) and lemma 10.6

$$\int_{\mathbb{R}} |P_j f(x)|\, dx \le C \int_{\mathbb{R}} \left( \int_{\mathbb{R}} 2^j W\left(2^j \frac{|x-y|}{2}\right) |f(y)|\, dy \right) dx$$

$$= C \int_{\mathbb{R}} |f(y)| \int_{\mathbb{R}} 2^j W\left(2^j \frac{|x-y|}{2}\right) dx \Big)\, dy$$

$$= 4C \|W\|_{L^1([0,\infty))} \|f\|_1$$

We have shown that the $P_{V_j}$ are bounded from $L^1(\mathbb{R})$ onto $L^1(\mathbb{R})$ and from $L^\infty(\mathbb{R})$ into $L^\infty(\mathbb{R})$. By the Riesz-Thorin interpolation theorem, $P_{V_j}$ is bounded from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$, $1<p<\infty$, with bound $4C\|W\|_{L^1([0,\infty))}$.

QED

**Remark.** Similar results as the one in Proposition 10.7 hold for the operators $P_{W_j}$ and $S^{\sigma}_{j,k}$ defined in (10.7) and (10.8).

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**Lemma 10.8** Assume that the scaling function $\varphi$ satisfies that $\hat{\varphi}(0)=1$. ~~(Most scaling functions satisfy this condition, but not the Shannon scaling function).~~ Assume also that $\varphi$ has a radial decreasing $L^1$-majorant. Then,

(a) $\sum_{k\in\mathbb{Z}} \varphi(x+k) = 1$ for almost every $x\in\mathbb{R}$

(b) $\int_{\mathbb{R}} 2^j K_\varphi(2^j x, 2^j y)\,dy = 1 \quad \forall j\in\mathbb{Z}$, for a.e. $x\in\mathbb{R}$

**Proof.** Since $\varphi\in L^1(\mathbb{R})$, the sum in (a) is welldefined. Then,

$$A(x) = \sum_{k\in\mathbb{Z}} \varphi(x+k)$$

is a periodic function on $\mathbb{R}$ and integrable in $[0,1)$ since

$$\int_0^1 |A(x)|dx = \int_0^1 \Big|\sum_{k\in\mathbb{Z}} \varphi(x+k)\Big|dx \le \sum_{k\in\mathbb{Z}} \int_k^{k+1} |\varphi(x)|\,dx$$

$$= \|\varphi\|_1 < \infty.$$

Let us compute the Fourier coefficients of $A$:

$$\hat{A}(\ell) = \int_0^1 A(x)\,e^{-2\pi i \ell x}dx = \int_0^1 \Big(\sum_{k\in\mathbb{Z}} \varphi(x+k)\Big)\,e^{-2\pi i \ell x}$$

$$= \sum_{k\in\mathbb{Z}} \int_k^{k+1} \varphi(x)\,e^{-2\pi i \ell x}dx = \int_{\mathbb{R}} \varphi(x)\,e^{-2\pi i \ell x}dx$$

$$= \hat{\varphi}(\ell)$$

Observe that $\hat{\varphi}$ is continuous because $\varphi\in L^1(\mathbb{R})$ and it makes good sense to write $\hat{\varphi}(\ell)$.

By Proposition 2.17 of Chapter 2 in Hernández–Weiss, $\hat{\varphi}(\ell)=0$ for all $\ell \in \mathbb{Z}$, $\ell \neq 0$. Thus $\hat{A}(\ell) = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell \neq 0 \end{cases}$. This implies that $A(x) = 1$ a.e.

To prove (b):

$$1 = \overline{\hat{\varphi}(0)} = \int_{\mathbb{R}} \overline{\varphi(z)}\, dz \overset{(a)}{=} \Big(\sum_{k\in\mathbb{Z}} \varphi(2^j x - k)\Big) \int_{\mathbb{R}} \overline{\varphi(z)}\, dz$$

$$= \sum_{k\in\mathbb{Z}} \varphi(2^j x - k)\, 2^j \int_{\mathbb{R}} \overline{\varphi(2^j y - k)}\, dy \overset{LDCT}{=}$$

$$= \int_{\mathbb{R}} 2^j \Big(\sum_{k\in\mathbb{Z}} \varphi(2^j x - k)\, \overline{\varphi(2^j y - k)}\Big) dy = \int_{\mathbb{R}} 2^j K_\varphi(2^j x, 2^j y)\, dy$$

QED

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Let us now consider the operator

$$(T_j f)(x) = \int_{\mathbb{R}} 2^j K(2^j x, 2^j y) \{f(y) - f(x)\}\, dy \qquad (10.11)$$

where $K(x,y) \le C\, W\left(\frac{|x-y|}{2}\right)$ and $W$ is a radial decreasing $L^1$-majorant as in Definition 10.5

**Theorem 10.9** For the operators $T_j$ defined in (10.10) we have

(a) If $1 \le p < \infty$, then $\lim_{j\to\infty} \|T_j f\|_p = 0 \quad \forall f \in L^p(\mathbb{R})$

(b) $\lim_{j\to\infty} \|T_j f\|_\infty = 0$ for all bounded unif. continuous functions $f$

**Proof** To prove (a) choose $f \in L^p(\mathbb{R})$, $1 \le p < \infty$. A change of variables give

$$|T_j f(x)| \le C \int_{\mathbb{R}} 2^j W\left(2^j \frac{|x-y|}{2}\right) |f(y) - f(x)|\, dy$$

$$= C \int_{\mathbb{R}} 2^j W\left(2^j \frac{|t|}{2}\right) |f(x+t) - f(x)|\, dt. \qquad (10.12)$$

By Minkowski's inequality for integrals,

$$\|T_j f\|_p \leq C\int_{\mathbb{R}} 2^j W\left(\frac{2^j|t|}{2}\right)\left(\int_{\mathbb{R}} |f(x-t)-f(x)|^p dx\right)^{1/p} dt$$

$$= C\int_{\mathbb{R}} W\left(\frac{|s|}{2}\right)\left(\int_{\mathbb{R}} |f(x-2^{-j}s)-f(x)|^p dx\right)^{1/p} ds$$

The expression

$$\left(\int_{\mathbb{R}} |f(x-h)-f(x)|^p dx\right)^{1/p} := \omega_p(h)$$

is the $L^p$-modulus of continuity of $f$, and satisfies $\omega_p(h)\to 0$ as $h\to 0$ if $1\leq p<\infty$. By the Lebesgue dominated convergence theorem (the integral is dominated by the $L^1$-function $2\|f\|_p W(\frac{|s|}{2})$) we obtain

$$0\leq \lim_{j\to\infty}\|T_j f\|_p \leq C\int_{\mathbb{R}} W\left(\frac{|s|}{2}\right)\left(\lim_{j\to\infty}\omega_p^p(2^{-j}s)\right)ds = 0$$

To prove (b), let $f\in L^\infty(\mathbb{R})$ be uniformly continuous. Then

$$\omega_\infty(t) := \sup_{x\in\mathbb{R}} |f(x-t)-f(x)| \longrightarrow 0 \quad \text{as } t\to 0$$

uniformly in $x$. From (10.12)

$$\|T_j f\|_\infty \leq C\int_{\mathbb{R}} 2^j W\left(\frac{2^j|t|}{2}\right)\omega_\infty(t)\,dt$$

$$= C\int_{\mathbb{R}} W\left(\frac{|s|}{2}\right)\omega_\infty(2^{-j}s)\,dt$$

Since $\omega_\infty$ is bounded by $2\|f\|_\infty$ and $W\in L^1([0,1))$, the proof is finished by applying the Lebesgue dominated convergence theorem. Q.E.D

**Corollary 10.10** Suppose $\varphi$ is a scaling function of an MRA with $\hat{\varphi}(0)=1$ and a radial decreasing $L^1$-majorant $W$.

(a) If $1\le p<\infty$, then $\lim_{j\to\infty}\|P_{V_j}f-f\|_p=0$ for all $f\in L^p(\mathbb{R})$

(b) $\lim_{j\to\infty}\|P_{V_j}f-f\|_\infty=0$ for all $f\in L^\infty(\mathbb{R})$ uniformly continuous

Proof. By Lemma 10.8, item (b),

$$P_{V_j}f(x)-f(x)=\int_{\mathbb{R}}2^jk_\varphi(2^jx,2^jy)\{f(y)-f(x)\}\,dy$$

wich coincides with $T_jf$ with $k=k_\varphi$. The Corollary follows from Theorem 10.9. QED

Exercise 10.F. Assume that the scaling function of an MRA satisfies $\hat{\varphi}(0)=1$. It is known that $\psi$ is an o.n. wavelet $\hat{\psi}(0)=\int_{\mathbb{R}}\psi(x)\,dx=0$. Assume that $\varphi$ and $\psi$ have a radial decreasing $L^1$-majorant. Use Corollary 10.10 and Theorem 10.9 to show that if $1\le p<\infty$, then

$$\lim_{J\to\infty}\|S^\sigma_{J,k}f-f\|_p=0\quad\forall k=1,2,3,\dots,\ \forall f\in L^p(\mathbb{R})$$

where $S^\sigma_{J,k}f$ is the partial sums operator defined in (10.8)

S/ By Lemma 10.5 and $\int_{\mathbb{R}}\psi(x)\,dx=0$,

$$S^\sigma_{J,k}f(x)-f(x)=(P_{V_J}f(x)-f(x))+\int_{\mathbb{R}}\Big\{\sum_{m=1}^{k}\psi_{J,\sigma(m)}(x)\overline{\psi_{J,\sigma(m)}(y)}\Big\}(f(y)-f(x))\,dy$$

$$:=(P_{V_J}f(x)-f(x))+\int_{\mathbb{R}}2^JK^{\sigma,k}(2^Jx,2^Jy)(f(y)-f(x))\,dy$$

By Corollary 10.10, item (a), $\lim_{j\to\infty} \|P_{V_j} f - f\| = 0$. Since $|k^{\sigma,k}(x,y)| \le c\, W\left(\frac{|x-y|}{2}\right)$, by item (a) of Theorem 10.9, the result is proved.