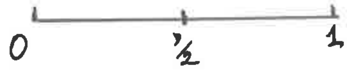
9. HAAR SYSTEM IN $L^p([0,1])$ SPACES, $1 \le p < \infty$

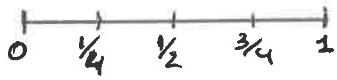
In the interval $[0,1]$ consider the following families of intervals

$\mathcal{D}_0 = \{ [0,1) \doteq I_{0,1} \}$

$\mathcal{D}_1 = \{ I_{1,1} = [0, \frac{1}{2}), \; I_{1,2} = [\frac{1}{2}, 1) \}$



$\mathcal{D}_2 = \{ I_{2,1} = [0, \frac{1}{4}), \; I_{2,2} = [\frac{1}{4}, \frac{1}{2}), \; I_{2,3} = [\frac{1}{2}, \frac{3}{4}), \; I_{2,4} = [\frac{3}{4}, 1) \}$



and for $j \ge 3$, $j \in \mathbb{N}$

$$\mathcal{D}_j = \{ I_{j,k} = [\frac{k-1}{2^j}, \frac{k}{2^j}) : k = 1, 2, \dots, 2^j$$

Each interval in $\mathcal{D}_j$ has measure $\frac{1}{2^j}$. The collection

$$\mathcal{D} = \bigcup_{j=0}^{\infty} \mathcal{D}_j$$

is called the set of dyadic intervals in $[0,1]$. The left end-points of the diadic intervals is the set of diady integers of $[0,1)$

$$\{ \frac{k-1}{2^j}, \; j = 0, 1, 2, \dots, \; k = 1, 2, \dots 2^j \}$$

which is dense in $[0,1)$.

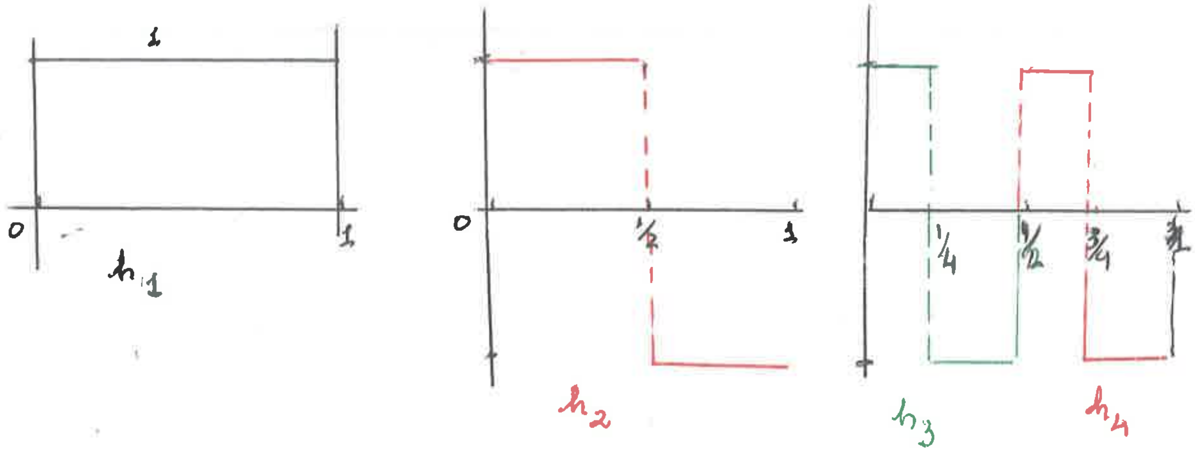
We order this intervals as follows:

$$I_{0,1}, \; \underbrace{I_{1,1}, I_{1,2}}_{\mathcal{D}_1}, \; \underbrace{I_{2,1}, I_{2,2}, I_{2,3}, I_{2,4}}_{\mathcal{D}_2}, \dots$$

Def 9.1 The Haar system is the sequence of functions $\{h_n\}_{n=1}^{\infty}$ defined on $[0,1)$ by $h_1 = 1$ and for $n = 2^j + k$ where $j = 0, 1, 2, \dots$ and $k = 1, 2, \dots, 2^j$

$$h_n(t) = \chi_{[\frac{2k-2}{2^{j+1}}, \frac{2k-1}{2^{j+1}})}(t) - \chi_{[\frac{2k-1}{2^{j+1}}, \frac{2k}{2^{j+1}})}(t) = \begin{cases} 1 & \text{if } \frac{2k-2}{2^{j+1}} \le t < \frac{2k-1}{2^{j+1}} \\ -1 & \text{if } \frac{2k-1}{2^{j+1}} \le t < \frac{2k}{2^{j+1}} \\ 0 & \text{otherwise} \end{cases}$$

The support of h_n, with $n=2^j+k$, is $\left[\frac{2k-2}{2^{j+1}}, \frac{2k}{2^{j+1}}\right) =$
$=\left[\frac{k-1}{2^j}, \frac{k}{2^j}\right) = I_{j,k}$.



These functions are normalized in $L^\infty([0,1])$.

Exercise 9.A. Show that for $1 \le p < \infty$, and $n = 2^j + k$,

$$\|h_n\|_p = |I_{j,k}|^{1/p} = 2^{-j/p}$$

S/ $$\|h_n\|_p^p = \int_0^1 |h_n(t)|\,dt = \int_{I_{j,k}} 1\,dt = |I_{j,k}| = 1/2^j$$

Proposition 9.2 The Haar system $\mathcal{H} (= \{h_n\}_{n=1}^\infty$ is dense in $L^p([0,1))$, that is $\overline{\text{span}}\{h_n\}_{n\ge1}^\infty = L_p([0,1])$, $1 \le p < \infty$

P/ From measure theory and the fact that dyadic points are dense in $[0,1]$, any $f \in L^p([0,1))$, $1 \le p < \infty$, can be approximated in $L^p([0,1])$ by constant functions in dyadic intervals of the same length. Precisely f can be approximated by

$$\sum_{k=1}^{2^j} 2^j \left(\int_{I_{j,k}} f(t)\,dt\right) \chi_{I_{j,k}}$$

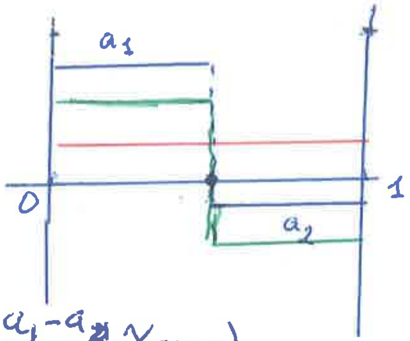
where the approximated value of f in the interval $I_{j,k}$ is obtained by the mean value theorem.

We need to show that each function of the form

$$f_j(t) = \sum_{k=1}^{2^j} a_k \chi_{I_{j,k}}(t), \qquad (9.1)$$

piecewise constant on the dyadic intervals of $\mathcal{D}_j$, can be written using the functions in the Haar system.

Let us start with $j=1$. Let



$$f_1 = a_1 \chi_{I_{1,1}} + a_2 \chi_{1,2}$$

$$= a_1 \chi_{[0,1/2)} + a_2 \chi_{[1/2,1)}$$

Then

$$f_1 = \frac{a_1+a_2}{2} \chi_{[0,1)} + \left(\frac{a_1-a_2}{2} \chi_{[0,1/2)} - \frac{a_1-a_2}{2} \chi_{[1/2,1)} \right)$$

$$= \frac{a_1+a_2}{2} h_1 + \frac{a_1-a_2}{2} h_2 \qquad (9.2)$$

which is a linear combination of two elements of the Haar system.

The same procedure works for a piecewise constant function on dyadic intervals of $\mathcal{D}_j$ such as (9.1). We can write

$$f_j = \sum_{k=1}^{2^j} a_k \chi_{I_{j,k}} = \sum_{k=1}^{2^j} a_k \chi_{[\frac{k-1}{2^j}, \frac{k}{2^j})} =$$

$$= \frac{a_1+a_2}{2} \chi_{[0, \frac{1}{2^{j-1}})} + \frac{a_1-a_2}{2} \left(\chi_{[0,\frac{1}{2^j})} - \chi_{[\frac{1}{2^j}, \frac{2}{2^j})} \right)$$

$$+ \frac{a_3+a_4}{2} \chi_{[\frac{1}{2^{j-1}}, \frac{2}{2^{j-1}})} + \frac{a_3-a_4}{2} \left(\chi_{[\frac{2}{2^j}, \frac{3}{2^j})} - \chi_{[\frac{3}{2^j}, \frac{4}{2^j})} \right)$$

$$\vdots$$

$$+ \frac{a_{2^j-1}+a_{2^j}}{2} \chi_{[\frac{2^{j-1}-1}{2^{j-1}}, 1)} + \frac{a_{2^j-1}-a_{2^j}}{2} \left(\chi_{[\frac{2^j-2}{2^j}, \frac{2^j-1}{2^j})} - \chi_{[\frac{2^j-1}{2^j}, 1)} \right)$$

$$= \frac{a_1+a_2}{2}\chi_{[0,\frac{1}{2^{j-1}})} + \frac{a_1-a_2}{2} h_{2^{j-1}+1}$$

$$+ \frac{a_3+a_4}{2}\chi_{[\frac{1}{2^{j-1}},\frac{2}{2^{j-1}})} + \frac{a_1-a_2}{2} h_{2^{j-1}+2}$$

$$\vdots$$

$$+ \frac{a_{2^j-1}+a_{2^j}}{2}\chi_{[\frac{2^{j-1}-1}{2^{j-1}},1)} + \frac{a_{2^j-1}-a_{2^j}}{2} h_{2^{j-1}+2^{j-1}}$$

We have written a piecewise constant function on dyadic intervals of $\mathcal{D}_j$ as a piecewise constant function, say it f_{j-1}, on dyadic intervals of $\mathcal{D}_{j-1}$ plus a linear combination of the Haar system elements of the form h_n with $n=2^{j-1}+k$ and $k=1,2,..,2^{j-1}$.

We can repeat this procedure with f_{j-1} to write it as a function f_{j-2} piecewise constant on dyadic intervals of $\mathcal{D}_{j-2}$ plus a linear combination of Haar system elements of the form h_n with $n=2^{j-2}+k$, $k=1,2,..,2^{j-2}$.

Continuing in this form $j-1$ times we arrive as a function f_1 such as (9.2) plus a linear combination of Haar system elements of the form h_n with

$$n=2^{j-\ell}+k,\quad \ell=1,2,..,j,\quad k=1,2,..,2^{j-\ell},$$

which in turn can be written as a linear combination of Haar system elements. This completes the proof. QED

Exercise 9.B Write $\chi_{[0,\frac14)}$ and $\chi_{[\frac12,\frac34)}$ as linear combination of elements of the Haar system.

S/ $\chi_{[0,\frac14)} = \frac14 h_1 + \frac14 h_2 + \frac12 h_3$

$\chi_{[\frac12,\frac34)} = \frac14 h_1 - \frac14 h_2 + \frac12 h_4$

We want to prove that the Haar system of Definition 9.1 is a Schauder basis for $L^p([0,1])$, $1 \le p < \infty$. We will use Proposition 6.12.

Theorem 9.3.

The Haar system (in the given order) is a monotone Schauder basis for $L^p([0,1])$, $1 \le p < \infty$.

Proof Item (i) of Prop 6.12 is clear. Item (iii) has been shown in Prop. 9.2. It remains to prove item (ii) of Prop 6.12 for $k=1$.

Exercise 9. C Show that if $x, y \in \mathbb{R}$, $x \ge 0, y \ge 0$, for all $1 \le p < \infty$, $(x+y)^p \le 2^{p-1}(x^p + y^p)$

S/ If $p=1$ it is an equality. If x or y are zero, the inequality is trivial because $1 \le 2^{p-1}$, $1 \le p < \infty$
Divide by y. Then one has to prove that

$$(x+1)^p \le 2^{p-1}(x^p+1) \quad , x \ge 0$$

Consider $\varphi(x) = \dfrac{2^{p-1}(x^p+1)}{(x+1)^p}$, $x \ge 0$, that satisfies $\varphi(1) = 1$. Show, by elementary calculus that φ has a global minimum at $x=1$.

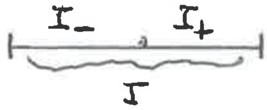
Let $f(t) = \sum_{n=1}^{m} a_n h_n(t)$ and $g(t) = \sum_{n=1}^{m+1} a_n h_n(t) = f(t) + a_{m+1} h_{m+1}(t)$.

Denote by I the support of h_{m+1}. The function f is constant in I because I only intersects dyadic intervals of double the length of I. Let $f(t) = b$ if $t \in I$. Then

$$\|g\|_p^p = \int_0^1 |f(t) + a_{m+1} h_{m+1}(t)|^p dt = \int_{[0,1)\setminus I} |f(t)|^p dt +$$

$$+ \int_I |b + a_{m+1} h_{m+1}(t)|^p dt$$

Denote by I_- the left-half interval of I and by I_+ the right half interval of I. Then



$$\|g\|_p^p = \int_{[0,1)\setminus I} |f(t)|^p dt + \int_{I_-} |b+a_{m+1}|^p dt + \int_{I_+} |b-a_{m-1}|^p dt$$

$$= \int_{[0,1)\setminus I} |f(t)|^p dt + \frac{|I|}{2} \left(|b+a_{m+1}|^p + |b-a_{m+1}|^p\right)$$

By Exercise 9.C

$$\|g\|_p^p \geq \int_{[0,1)\setminus I} |f(t)|^p dt + \frac{|I|}{2}\frac{1}{2^{p-1}} \left(|b+a_{m+1}| + |b-a_{m+1}|\right)^p$$

$$\geq \int_{[0,1]\setminus I} |f(t)|^p dt + \frac{|I|}{2^p}(2|b|)^p$$

$$= \int_{[0,1)\setminus I} |f(t)|^p dt + \int_I |f(t)|^p dt = \|f\|_p^p$$

By Prop 6.12 this shows that the Haar system is a Schauder basis for $L^p([0,1])$, $1 \leq p < \infty$

We know from Exercise 9.A that $\|h_n\|_p = 2^{-j/p}$, $n = 2^j + k$, $j = 0, 1, 2, \dots$, $k = 1, 2, \dots, 2^j$. Therefore, if we define for $1 \leq p < \infty$

$$h_n^{(p)}(t) := 2^{j/p} h_n(t) \quad , \quad n = 2^j + k$$

the collection $\mathcal{H}^{(p)} := \{h_n^{(p)}\}_{n=1}^{\infty}$ (with the order described before Def 9.1) is a normalized Schauder basis of $L^p([0,1])$, $1 \leq p < \infty$.

Exercise 9.C Show that for $p=2$, the set $\mathcal{H}^{(2)} = \{h_n^{(2)}\}_{n=1}^{\infty}$ is an orthonormal system in $L^2([0,1])$.

S/ Let $m < n$. If the dyadic sets that support h_n and

h_m are disjoint, clearly $\int_0^1 h_n^{(2)}(t)\, h_m^{(2)}(t)\,dt = 0$. If they have non empty intersection and call $I_n = \operatorname{supp} h_n$, $I_m = \operatorname{supp} h_m$, I_m is contained either in the left half I_n^- or in the right half I_n^+ of I_m. Since $h_n^{(2)}$ is constant in I_n^- and I_n^+ it follows that

$$\langle h_n^{(2)}, h_m^{(2)} \rangle = c\int_{I_m} h_m(t)\,dt = c\int_{I_n^-} 1\,dt - c\int_{I_n^+} 1\,dt = 0.$$

Exercise 9.C and Proposition 9.2 shows that $\mathcal{H}^{(2)} = \{h_n^{(2)}\}_{n=1}^{\infty}$ is an <u>orthonormal basis</u> of the Hilbert space $L^2([0,1])$. By Plancherel identity (theorem 5.11) for all $f \in L^2([0,1])$ (*)

$$\|f\|_2^2 = \sum_{n=1}^{\infty} |\langle f, h_n^{(2)}\rangle|^2 = |\langle f, 1\rangle|^2 + \sum_{j=0}^{\infty}\sum_{k=1}^{2^j} |\langle f, h_{2^j+k}^{(2)}\rangle|^2 \qquad (9.3)$$

<u>Proposition 9.4</u>. the Haar system $\mathcal{H}^{(2)} = \{h_n^{(2)}\}_{n=1}^{\infty}$ is an <u>unconditional basis</u> of $L^2([0,1])$ with unconditional constant 1

P/ By item (iii) of Theorem 7.8 it is enough to prove that the operator S_ε is bounded, where $\varepsilon = \{\varepsilon_n\}$, $\varepsilon_n = \pm 1$. This follows from (9.3) since

$$\|S_\varepsilon(f)\|^2 = \Big\|\sum_{n=1}^{\infty} \varepsilon_n \langle f, h_n^{(2)}\rangle h_n^{(2)}\Big\|^2 = \sum_{n=1}^{\infty} |\varepsilon_n \langle f, h_n^{(2)}\rangle|^2 =$$

$$= \sum_{n=1}^{\infty} |\langle f, h_n^{(2)}\rangle|^2 = \|f\|^2.$$

QED.

(*) f can be written as $f = \sum_{n=1}^{\infty} \langle f, h_n^{(2)}\rangle h_n^{(2)}$ with convergence in $L^2([0,1])$

Formula (9.3) provides a characterization of functions in $L^2([0,1])$ using the coefficients $\{<f, h_n^{(2)}>\}_{n=1}^{\infty}$. Littlewood-Paley theory provides formulas to decide when a function f is in $L^p([0,1])$ in terms of the coefficients $\{<f, h_n^{(2)}>\}_{n=1}^{\infty}$ but the expression is more complicated than (9.3).

Given a function $f \in L^1([0,1])$ define, for $t \in [0,1]$

$$(Wf)(x) = \left\{ |<f,1>|^2 + \sum_{j=0}^{\infty} \sum_{k=1}^{2^j} |<f, h_{2^j+k}^{(2)}>|^2 \, 2^j \chi_{I_{j,k}}(x) \right\}^{1/2} \qquad (9.4)$$

Littlewood-Paley Theory provides the following result

Theorem 9.5 Let $1<p<\infty$. There exists two constants $0<A_p \le B_p < \infty$ such that for all $f \in L^p([0,1])$

$$A_p \|f\|_p \le \|W(f)\|_p \le B_p \|f\|_p .$$

Corollary 9.6. Let $1<p<\infty$. The Haar basis $\{h_n^{(2)}\}_{n=1}^{\infty}$ is an unconditional basis of $L^p([0,1])$.

P/ If $f = <f,h_1>h_1 + \sum_{j=0}^{\infty} \sum_{k=1}^{2^j} <f, h_{2^j+k}> h_{2^j+k}$, then for

$\varepsilon = \{\varepsilon_n\}_{n=1}^{\infty}$, $\varepsilon_n = \pm 1$,

$$S_\varepsilon f = \varepsilon_1 <f,h_1>h_1 + \sum_{j=0}^{\infty} \sum_{k=1}^{2^j} \varepsilon_{j,k} <f, h_{2^j+k}> h_{2^j+k} .$$

By theorem 9.5 and (9.4)

$$\|S_\varepsilon f\|_p \le \frac{1}{A_p} \|W(S_\varepsilon f)\|_p = \frac{1}{A_p} \|W(f)\| \le \frac{B_p}{A_p} \|f\|_p .$$

This proves the result by item (iii) of Theorem 7.8

Before we continue, let us show that $(L^\infty([0,1]), \| \|_\infty)$ cannot have a Schauder basis. If $\mathcal{B}=\{f_n\}_{n=1}^\infty$ were a Schauder basis for $(L^\infty([0,1]), \| \|_\infty)$, the finite linear combinations of the form

$$\sum_{n=1}^{N} a_n f_n \quad , \quad a_n \in \mathbb{F} \; , \; N \in \mathbb{N}$$

are dense in $L^\infty([0,1], \| \|_\infty)$. Taking the a_n to be rational (real or complex), the set

$$\mathcal{D} = \left\{ \sum_{n=1}^{N} q_n f_n : q_n \text{ rational}, \; N \in \mathbb{N} \right\}$$

is countable and dense in $(L^\infty([0,1]), \| \|_\infty)$. Therefore, $(L^\infty([0,1], \| \|_\infty)$ would be a separable Banach space. But we will show in the next result that $(L^\infty[0,1], \| \|_\infty)$ is <u>not</u> separable.

Proposition 9.10 The Banach space $(L^\infty([0,1]), \| \|_\infty)$ is <u>not</u> separable.

P/ For each $n \in \mathbb{N}$, let $I_n = [\frac{1}{n+1}, \frac{1}{n}]$. The intervals I_n are disjoint and $|I_n| = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0$. For each non empty set $C \subset \mathbb{N}$ define

$$f_C = \chi_{\underset{n \in C}{\cup} I_n} \in L^\infty([0,1])$$

If $D \subset \mathbb{N}$ and $D \neq C$, $\| f_C - f_D \|_\infty = 1$ porque there must exists an interval I_n in which one of the functions takes the value 1 and the other function takes de value cero.

If $\mathcal{D}$ were a dense set in $L^\infty([0,1])$, each open ball $B_{1/2}(f_C)$ must contain at least one point of $\mathcal{D}$. But $B_{\frac{1}{2}}(f_C) \cap B_{\frac{1}{2}}(f_D) = \emptyset$ if $C \neq D$ because $\| f_C - f_D \|_\infty = 1$. Therefore the number of elements of $\mathcal{D}$ is at least the number of elements of the family $\{ f_C : \emptyset \neq C \subset \mathbb{N} \}$. Since the parts of $\mathbb{N}$ is not a countable set, so $\{ f_C : \emptyset \neq C \subset \mathbb{N} \}$ is also not countable. Hence $\mathcal{D}$ cannot be countable.

We will give a proof of Corollary 9.6 due to D. Burkholder (A non linear partial differential equation and the unconditional constant of the Haar system in L^p, Bull Amer. Math Soc (N.S) 7(3) 591-595 (1982)). The proof is taken from section 6.1 of Albiac-Kalton book.

Remark 1: The original proof is due to Paley (R.E.A.C. Paley, A remarkable series of orthogonal functions, Proc. Lond. Math. Soc 34, 241-264 (1932))

Remark 2 The proof of ~~Al~~ Albiac-Kalton's book is taken from D. Burkholder (A proof of Pełczyński's conjecture on the Haar system, Stud. Math 91(1) 79-83 (1988))

We borrow from Albiac-Kalton's book the following lemma (page 142)

Lemma 9.11 Suppose $p>2$ and define a function φ on $\mathbb{R}^+\times\mathbb{R}^+$ by

$$\varphi(x,y)=(x+y)^{p-1}[(p-1)x-y]\ ,\quad x,y\geq 0$$

(a) The following inequality holds for all (x,y) with $x\geq 0$ and $y\geq 0$

$$(p-1)^p x^p - y^p \geq \frac{(p-1)^{p-1}}{p^{p-2}}\varphi(x,y)$$

(b) For all real numbers x,y,a and for $\varepsilon=\pm 1$

$$\varphi(|x+a|,|y+\varepsilon a|)+\varphi(|x-a|,|y-\varepsilon a|)\geq 2\varphi(|x|,|y|)$$

For $1<p<\infty$, define

$$p^*=\begin{cases}\dfrac{p}{p-1} & \text{if } p\leq 2\\ p & \text{if } p\geq 2\end{cases}$$

Notice that $\frac{p}{p-1}=q \Leftrightarrow \frac{1}{p}+\frac{1}{q}=1$.

Theorem 9.12. Suppose $1<p<\infty$. The Haar basis $\mathcal{H}=\{h_n\}_{n=1}^{\infty}$ given in Def 9.1 is an unconditional basis of $L^p([0,1])$ with unconditional constant at most p^*-1.

P/ By Exercise 7, it is enough to prove that for all $N\in\mathbb{N}$

$$\left\|\sum_{n=1}^{N}\varepsilon_n a_n h_n\right\|_p \le (p^*-1)\left\|\sum_{n=1}^{N} a_n h_n\right\|_p \qquad (9.5)$$

for any $a_1, a_2, \dots, a_N \in \mathbb{F}$ and any choice of signs $\varepsilon_1, \dots, \varepsilon_N$.

Suppose $p>2$. Then $p^*=p$. By lemma 9.11 (a) it is enough to show that

$$\int_0^1 \varphi\left(\left|\sum_{n=1}^{N} a_n h_n(t)\right|, \left|\sum_{n=1}^{N}\varepsilon_n a_n h_n(t)\right|\right) dt \ge 0 \qquad (9.6)$$

for all $N\in\mathbb{N}$, all $a_1, a_2, \dots, a_N\in\mathbb{F}$ and all $\{\varepsilon_n\}_{n=1}^{N}$, $\varepsilon_n=\pm1$.

Indeed, if (9.6) holds, by lemma 9.11 (a),

$$\int_0^1\left((p-1)^p\left|\sum_{n=1}^{N} a_n h_n(t)\right|^p - \left|\sum_{n=1}^{N}\varepsilon_n a_n h_n(t)\right|\right) dt \overset{\text{Lemma 9.11}}{\ge}$$

$$\frac{(p-1)^{p-1}}{p^{p-2}}\int_0^1 \varphi\left(\left|\sum_{n=1}^{N} a_n h_n(t)\right|, \left|\sum_{n=1}^{N}\varepsilon_n a_n h_n(t)\right|\right)dt \overset{(9.6)}{\ge} 0$$

This implies

$$(p-1)^p\int_0^1\left|\sum_{n=1}^{N} a_n h_n(t)\right|^p dt \ge \int_0^1\left|\sum_{n=1}^{N}\varepsilon_n a_n h_n(t)\right|^p dt$$

which is (9.5) with $p^*=p$ (since $p>2$).

We prove (9.6) by induction on N. If $N=1$, use lemma 9.11 (b) with $x=y=0$ to obtain

$$\int_0^1\varphi(|a_1|, |a_1|)\,dt = \varphi(|a_1|,|a_1|) \ge \varphi(0,0) = 0.$$

Suppose $N \geq 2$ and assume that (9.6) holds for $(N-1)$ tuples.
Let $\{a_n\}_{n=1}^{N} \subset \mathbb{F}$ and $\{\varepsilon_n\}_{n=1}^{N}$, $\varepsilon_n = \pm 1$. Let

$$f_{N-1} = \sum_{n=1}^{N-1} a_n h_n \quad , \quad g_{N-1} = \sum_{n=1}^{N-1} \varepsilon_n a_n h_n$$

$$f_N = f_{N-1} + a_N h_N \quad , \quad g_N = g_{N-1} + \varepsilon_N a_N h_N$$

$$\tilde{f}_N = f_{N-1} - a_N h_N \quad , \quad \tilde{g}_N = g_{N-1} - \varepsilon_N a_N h_N$$

Observe that h_N is supported on a dyadic interval on which f_{N-1} and g_{N-1} are constant. Also h_N takes oposite values in intervals of the same length. Therefore

$$J := \int_0^1 \varphi(|f_N(t)|, |g_N(t)|)\,dt = \int_0^1 \varphi(|\tilde{f}_N(t)|, |\tilde{g}_N(t)|)\,dt$$

By lemma 9.11 (b)

$$J = \int_0^1 \frac{1}{2}\left[\varphi(|f_N(t)|, |g_N(t)|) + \varphi(|\tilde{f}_N(t)|, |\tilde{g}_N(t)|)\right] dt$$

$$\geq \int_0^1 \varphi(|f_{N-1}(t)|, |g_{N-1}(t)|)\,dt \geq 0$$

by our induction hypothesis. Therefore

$$J = \int_0^1 \varphi(|f_N(t)|, |g_N(t)|)\,dt \geq 0$$

as we wanted to prove.

This shows the Theorem for $p > 2$. The case $p = 2$ is true because $L^2([0,1])$ is a Hilbert space and $\{h_n\}_{n=1}^{\infty}$ is an orthogonal basis of $L^2([0,1])$. (Exercise 7.D).

For the case $1 < p < 2$ we use the duality between $L^p([0,1])$ and $L^q([0,1])$, $\frac{1}{p} + \frac{1}{q} = 1$. (See Theorem 4.5). Observe that $2 < q < \infty$.

Let $f_N = \sum_{n\geq 1}^{N} a_n h_n$ and $g_N = \sum_{n\geq 1}^{N} \varepsilon_n a_n h_n$. By duality

$$\|g_N\|_p = \sup_{\|g_N'\|_q = 1} \int_0^1 g_N(t)\, \overline{g_N'(t)}\, dt \qquad (9.7)$$

For g_N' such that $\|g_N'\|_q = 1$ write $\overline{g_N'} = \sum_{n=1}^{N} b_n h_n$ for some $\{b_n\}_{n\geq 1}^{N} \subset \mathbb{F}$. Let $f_N' = \sum_{n\geq 1}^{N} \varepsilon_n b_n h_n$. Then

$$\int_0^1 g_N(t)\, \overline{g_N'(t)}\, dt = \int_0^1 \Big(\sum_{n\geq 1}^{N} \varepsilon_n a_n h_n(t)\Big)\Big(\sum_{n\geq 1}^{N} b_n h_n(t)\Big)\, dt$$

$$= \int_0^1 \Big(\sum_{n\geq 1}^{N} a_n h_n(t)\Big)\Big(\sum_{n\geq 1}^{N} \varepsilon_n h_n(t)\Big)\, dt = \int_0^1 f_N(t)\, f_N'(t)\, dt$$

Hölder's

$$\leq \|f_N\|_p \, \|f_N'(t)\|_q \qquad (9.8)$$

Since $q>2$, and we have already proved (9.5) for $q>2$, we have $\|f_N'\|_q \leq (q-1)\|g_N'\|_q$. From (9.7) and (9.8) we deduce

$$\|g_N\|_p \leq \sup_{\|g_N'\|_q = 1} \|f_N\|_p (q-1) \|g_N'\|_q = (p^* - 1)\|f_N\|_p$$

since $q = \frac{p}{p-1} = p^*$ when $p \leq 2$.

Remark In Theorem 9.12, the constant (p^*-1) is the best possible. This was proved by D. Burkholder.

<u>Exercise 9.D</u> For $n = 2^j + k$, $j = 0, 1, 2, \ldots$, $k = 1, 2, \ldots, 2^j$, define $\varphi_n(x) = 2^{j+1} \int_0^x h_n(t)\,dt$, where $\{h_n\}_{n=1}^{\infty}$ is the Haar system of Def 9.1.

Show that

$$\varphi_n(x) = \begin{cases} 2^{j+1}x - (2k-2) & \text{if } \frac{2k-2}{2^{j+1}} \le x \le \frac{2k-1}{2^{j+1}} \\ -2^{j+1}x + 2k & \text{if } \frac{2k-1}{2^{j+1}} \le x \le \frac{2k}{2^{j+1}} \\ 0 & \text{otherwise} \end{cases}$$

Draw the graphs of φ_2, φ_3 and φ_4.

S/ The support of $h_n = I_{j,k} = \left[\frac{2k-2}{2^{j+1}}, \frac{2k}{2^{j+1}}\right] = \left[\frac{k-1}{2^j}, \frac{k}{2^j}\right]$. Hence, if $x \le \frac{2k-2}{2^{j+1}}$, $\varphi_n(x) = 0$. For $\frac{2k-2}{2^{j+1}} \le x \le \frac{2k-1}{2^{j+1}}$,

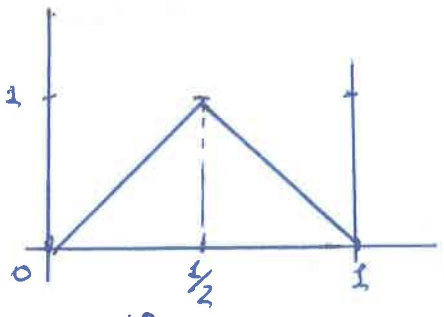
$$\varphi_n(x) = 2^{j+1} \int_{\frac{2k-2}{2^{j+1}}}^{x} 1\,dt = 2^{j+1}\left[x - \frac{(2k-2)}{2^{j+1}}\right] = 2^{j+1}x - (2k-2).$$

For $\frac{2k-1}{2^{j+1}} \le x \le \frac{2k}{2^{j+1}}$,

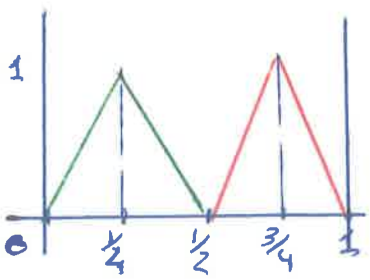
$$\varphi_n(x) = 2^{j+1}\left[\int_{\frac{2k-2}{2^{j+1}}}^{\frac{2k-1}{2^{j+1}}} dt - \int_{\frac{2k-1}{2^{j+1}}}^{x} dt\right] = 2^{j+1}\left[\frac{1}{2^{j+1}} - \left(x - \frac{2k-1}{2^{j+2}}\right)\right]$$

$$= 2^{j+1}x - 2k.$$

Since $\varphi_n\left(\frac{2k}{2^{j+1}}\right) = 0$, and $h_n(t) = 0$ when $n \ge \frac{2k}{2^{k+1}}$, it follows that $\varphi_n(x) = 0$ if $\frac{2k}{2^{j+2}} \le x \le 1$.



φ_2



φ_3 φ_4

Remarks Defining $\varphi_0(x)=1$ and $\varphi_1(x)=x$, the set $\{\varphi_n\}_{n=0}^{\infty}$ is the original basis of $C([0,1])$ published by Schauder