8. Exponential Bases of $L^p(\mathbb{T})$, $1 \le p < \infty$, and $C(\mathbb{T})$

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. For each $n \in \mathbb{Z}$, let $e_n : \mathbb{T} \to \mathbb{C}$ be the function $e_n(z) = z^n$. We can identify the space $C(\mathbb{T})$ of continuous complex-valued functions on $\mathbb{T}$ with the space $C_p([0,1])$ of continuous 1-periodic functions defined on $\mathbb{R}$. Writing $z = e^{2\pi i \theta}$ we shall use the notation $e_n(\theta) = e^{2\pi i n \theta}$.

Consider the sequence $\mathcal{E} = (e_0, e_1, e_{-1}, e_2, e_{-2}, \dots)$ in this particular order. We want to study if $\mathcal{E}$ is a basis of $C_p([0,1])$.

Note for the reader. $\mathcal{E}$ is not a basis of $(C_p([0,1]), \|\ \|_\infty)$. This is a classical result, since it is known that there exists a function in $C_p([0,1])$ whose Fourier series does not converge uniformly (see Y. Katznelson, An Introduction to Harmonic Analysis, 2nd Edition, Dover, 1976)

To keep in mind that we are dealing with periodic functions let us write $L^p_{per}([0,1])$ to be the space of complex-valued functions 1-periodic defined on $\mathbb{R}$.

Def 8.1 If $f \in L^1_{per}([0,1])$ and $k \in \mathbb{Z}$ the k^{th}-Fourier coefficient of f is

$$\hat{f}(k) = \int_0^1 f(x)\,\overline{e_k(x)}\,dx = \int_0^1 f(x)\, e^{-2\pi i k x}\,dx$$

When $f \in L^1_{per}([0,1])$,

$$|\hat{f}(k)| \le \int_0^1 |f(x)|\,dx = \|f\|_1 < \infty$$

The Fourier series of f is the formal series

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{2\pi i k x} \qquad (8.1)$$

We will be interested in the convergence of the series (8.1) in the sense of the convergence of the symetric partial sums.

For each $N \in \mathbb{N}$ let

$$S_N f(x) := \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x}, \quad x \in \mathbb{R}. \qquad (8.2)$$

Exercise 8.A Prove that $S_N f(x) = \int_0^1 f(t) D_N(x-t)\,dt$

where

$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi i k t} = \begin{cases} 2N+1 & \text{if } t=0 \\ \dfrac{\sin(2N+1)\pi t}{\sin \pi t} & \text{if } t \neq 0 \end{cases},$$

is the so called Dirichlet kernel

S/ $S_N f(x) = \sum_{k=-N}^{N} \left(\int_0^1 f(t) e^{-2\pi i k t} dt \right) e^{2\pi i k x}$

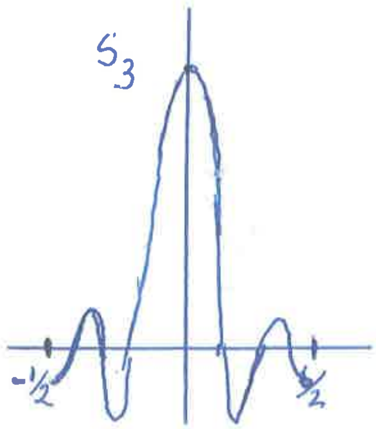
$$= \int_0^1 \left(\sum_{k=-N}^{N} e^{2\pi i k (x-t)} \right) f(t)\,dt = \int_0^1 f(t) D_N(x-t)\,dt$$

Now

$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi i k t} \overset{t\neq 0}{=} \frac{e^{2\pi i (N+1) t} - e^{-2\pi i N t}}{e^{2\pi i t} - 1} =$$

$$= \frac{e^{2\pi i (N+\frac{1}{2})t} - e^{-2\pi i (N+\frac{1}{2})t}}{e^{\pi i t} - e^{-\pi i t}} = \frac{\sin \pi(2N+1)t}{\sin \pi t}$$

and if $t=0$, $D_N(0) = \sum_{k=-N}^{N} 1 = 2N+1$.



$D_N(t)$ is 1-periodic and even.

Moreover,

$$\int_0^1 D_N(t)\,dt = \int_0^1 \left(\sum_{k=-N}^{N} e^{2\pi i k t} \right) dt$$

$$= \sum_{k=-N}^{N} \int_0^1 e^{2\pi i k t}\,dt = 1.$$

Warning Although $\int_0^1 D_N(t)\,dt = 1$, D_N does not belong to $L^1_{per}([0,1])$ because $\int_0^1 |D_N(t)|\,dt \approx \ln N$ (see Katznelson's book, Exercise 1 on page 59).

For $f \in L^1_{per}([0,1])$ define

$$\sigma_N f(x) := \frac{1}{N} \sum_{k=0}^{N-1} S_k f(x) \quad , x \in \mathbb{R}. \tag{8.3}$$

We have, by 1-periodicity,

$$\sigma_N f(x) = \frac{1}{N} \sum_{k=0}^{N-1} \int_0^1 f(x-t) D_k(t)\,dt = \int_0^1 f(x-t)\left[\frac{1}{N}\sum_{k=0}^{N-1} D_k(t)\right]dt$$

$$:= \int_0^1 f(x-t) F_N(t)\,dt$$

where $F_N(t) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(t)$ is called the Fejér kernel.

Observe that

$$\int_0^1 F_n(t)\,dt = \frac{1}{N} \sum_{k=0}^{N-1} \int_0^1 D_k(t)\,dt = 1 \tag{8.4}$$

Exercise 8.B The Fejér kernel is 1-periodic, even, and

$$F_N(t) = \begin{cases} \frac{1}{N}\left(\frac{\sin \pi N t}{\sin \pi t}\right)^2 & \text{if } t \in \mathbb{R} - \mathbb{Z} \\ N & \text{if } t \in \mathbb{Z} \end{cases}$$

S/ Enoug to prove the result for $t \in [-\frac{1}{2}, \frac{1}{2}]$ by 1-periodicity.

$$F_N(0) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(0) = \frac{1}{N}\left[1+3+5+\dots+(2N-1)\right] = \frac{1}{N}\,\frac{2N \cdot N}{2} = N$$

If $t \neq 0$,

$$F_N(t) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{\sin \pi(2k+1)t}{\sin \pi t} = \frac{1}{N}\,\frac{1}{\sin \pi t} \sum_{k=0}^{N-1} \sin \pi(2k+1)t \tag{8.5}$$

Use that $2\sin A \sin B = \cos(A-B) - \cos(A+B)$ with $A=(2k+1)\pi t$ and $B=\pi t$ to obtain that

$$\sum_{k=0}^{N-1} \sin \pi(2k+1)t = \frac{1}{2\sin \pi t} \sum_{k=0}^{N-1} \left[\cos 2k\pi t - \cos 2((k+1)\pi t)\right]$$

$$= \frac{1}{2\sin \pi t}\left[1 - \cos 2\pi t + \cos 2\pi t - \cos 4\pi t + \dots - \cos(2N\pi t)\right]$$

$$= \frac{1}{2\sin \pi t}\left[1 - \cos(2N\pi t)\right] = \frac{\sin^2 \pi N t}{\sin \pi t}$$

where in the last equality we have used $\sin^2 \alpha = \frac{1-\cos 2\alpha}{2}$. Replacing in (8.5) we conclude

$$F_N(t) = \frac{1}{N}\left(\frac{\sin \pi N t}{\sin \pi t}\right)^2.$$

Observe that $F_N(t) \geq 0$ for all $t \in \mathbb{R}$. By (8.4)

$$\int_0^1 |F_N(t)|\,dt = \int_0^1 F_N(t)\,dt = 1, \quad N = 1, 2, 3, \dots \quad (8.6)$$

Exercise 8.C. For every $0 < \delta < \frac{1}{2}$, $\lim_{N\to\infty} \int_{\delta<|t|<\frac{1}{2}} F_N(t)\,dt = 0$

S/ From Exercise 8.B

$$0 \leq \int_{\delta<|t|<\frac{1}{2}} F_N(t)\,dt \leq \frac{1}{N}\int_{\delta<|t|<\frac{1}{2}} \frac{1}{\sin^2 \pi t}\,dt \leq \frac{1}{N}\frac{1}{\sin^2 \pi\delta} \xrightarrow{N\uparrow\infty} 0.$$

$\boxed{\begin{array}{l}\textbf{Theorem 8.2 (A).}\text{ Let } f \in C_{per}([-\tfrac{1}{2},\tfrac{1}{2}]), \text{ then} \\ \qquad \lim_{N\to\infty} \|\sigma_N(f) - f\|_\infty = 0 \\ \text{(B) Let } f \in L^p_{per}([-\tfrac{1}{2},\tfrac{1}{2}]),\ 1 \leq p < \infty. \text{ Then} \\ \qquad \lim_{N\to\infty} \|\sigma_N(f) - f\|_p = 0\end{array}}$

Proof. (A) Since $[-\frac{1}{2},\frac{1}{2}]$ is compact, f is uniformly continuous. Given $\varepsilon>0$, let $\delta=\delta(\varepsilon)>0$ be such that

$$\sup_{|y|<\delta} |f(x-y)-f(x)| \le \varepsilon/2 \qquad (8.7)$$

for all $x\in[-\frac{1}{2},\frac{1}{2}]$. Then, since $\int_{-1/2}^{1/2} F_N(t)\,dt = 1$ by (8.6),

$$|\sigma_N(f)(x)-f(x)| = \left|\int_{-1/2}^{1/2} [f(x-t)-f(x)]\,F_N(t)\,dt\right| \overset{F_N(t)\ge 0}{\le}$$

$$\le \int_{|t|<\delta} |f(x-t)-f(x)|\,F_N(t)\,dt + \int_{\delta<|t|\le\frac{1}{2}} |f(x-t)-f(x)|\,F_N(t)$$

$$:= I_{\delta,N} + II_{\delta,N} \qquad (8.8)$$

By (8.7) and (8.6)

$$I_{\delta,N} \le \left(\sup_{|t|<\delta} |f(x-t)-f(x)|\right)\int_{-1/2}^{1/2} F_N(t)\,dt \le \varepsilon/2$$

By Exercise 8.B, there exist $N_0=N_0(\delta)=N_0(\varepsilon)\in\mathbb{N}$ such that

$$\int_{\delta<|t|\le\frac{1}{2}} F_N(t)\,dt \le \frac{\varepsilon}{4\|f\|_\infty}, \quad \forall N\ge N_0.$$

Hence

$$II_{\delta,N} \le 2\|f\|_\infty \int_{\delta<|t|\le 1/2} F_N(t)\,dt \le 2\|f\|_\infty \frac{\varepsilon}{4\|f\|_\infty} = \frac{\varepsilon}{2}, \quad N\ge N_0$$

Replacing these estimates in (8.8) and taking the supremum over all $x\in[-\frac{1}{2},\frac{1}{2}]$ we obtain $\|\sigma_N(f)-f\|_\infty<\varepsilon$ for all $N\ge N_0$, thus proves (A).

(B) Let $f\in L^p_{per}([-\frac{1}{2},\frac{1}{2}])$. Given $\varepsilon>0$, choose $g\in C_{per}([-\frac{1}{2},\frac{1}{2}])$ such that $\|f-g\|_p\le\varepsilon$. (This can be done because $C_{per}([-\frac{1}{2},\frac{1}{2}])$ is dense in $L^p_{per}([-\frac{1}{2},\frac{1}{2}])$, $1\le p<\infty$. Then

$$\|\sigma_N f - f\|_p \le \left\|\int_0^1 f(\cdot-t)F_N(t)\,dt - \int_0^1 g(\cdot-t)F_N(t)\,dt\right\|_p$$

$$\cdots + \left\| \int_0^1 g(\cdot - t) F_N(t)\,dt - g \right\|_p + \|g - f\|_p \overset{(8.4)}{=}$$

$$= \left\| \int_0^1 [f(\cdot - t) - g(\cdot - t)] F_N(t)\,dt \right\|_p + \left\| \int_0^1 (g(\cdot - t) - g(\cdot)) F_N(t)\,dt \right\|_p + \|g - f\|_p$$

Observe that

$$\int_0^1 (f(x-t) - g(x-t)) F_N(t)\,dt = F_N * (f-g)(t).$$

By Young's inequality

$$\| F_N * (f-g) \|_p \le \|F_n\|_1 \|f-g\|_p = \|f-g\|.$$

Also, by (Δ) there exist $N_0 \in \mathbb{N}$ s.t. for $N \ge N_0$

$$\left\| \int_0^1 (g(\cdot - t) - g(\cdot)) F_N(t)\,dt \right\|_p = \| \sigma_N(g) - g \|_p \le \|\sigma_N(g) - g\|_\infty < \varepsilon$$

Then

$$\|\sigma_N(f) - f\|_p \le 2\|f-g\| + \varepsilon \le 3\varepsilon, \quad \forall N \ge N_0.$$

QED

Corollary 8.3 The set $\mathcal{E} = \{ e_n(x) = e^{2\pi i n x} \}_{n \in \mathbb{Z}}$ is such that span $\mathcal{E}$ = span $\{ e_n \}_{n \in \mathbb{Z}}$ is dense on $(L^p(\mathbb{T}), \| \, \|_p)$ $1 \le p < \infty$, and on $(C(\mathbb{T}), \| \, \|_\infty)$.

P/ Observe that

$$\sigma_N f(x) = \frac{1}{N} \sum_{k=0}^{N-1} S_k f(x) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=-k}^{k} \hat{f}(j) e^{2\pi i j x}$$

is a finite linear combination of $e_0, e_1, e_{-1}, \dots, e_{N-1}, e_{-(N-1)}$.

Theorem 8.2 finishes the proof. QED.

It is clear that

$$\int_0^1 e_n(t)\overline{e_m(t)}\,dt = \int_0^1 e^{2\pi i(n-m)t}\,dt = \begin{cases} 1 \text{ if } n=m \\ 0 \text{ if } n\neq m \end{cases}$$

Therefore $\mathcal{E}$ is an orthonormal system in $L^2(\mathbb{T})$. By Corollary 8.3, $\mathcal{E}$ is dense in $L^2(\mathbb{T})$. Hence, $\mathcal{E}$ is an orthonormal basis of the Hilbert space $(L^2(\mathbb{T}), \|\ \|_2)$. Planchorel identity (Theorem 5.11) gives

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \quad , \quad f\in L^2(\mathbb{T}) \qquad (8.5)$$

By Example 6.9 it is a basis of $(L^2(\mathbb{T}), \|\ \|_2)$ in the sense of Def 6.3 with is monotone, that is $K_b(\mathcal{E})=1$. By Exercise 7.D, $\mathcal{E}$ is an unconditional basis of $L^2(\mathbb{T}), \|\ \|_2)$.

To study the convergence of Fourier series in $L^p(\mathbb{T})$ with $1\leq p<\infty$, $p\neq 2$, and $(C(\mathbb{T}), \|\ \|_\infty)$ we need the following result.

Proposition 8.4 The following statements are equivalent for any $1\leq p\leq\infty$

(i) For every $f\in L^p(\mathbb{T})$ (or $f\in C(\mathbb{T})$ if $p=\infty$) we have

$$\lim_{N\to\infty} \|S_N f - f\|_p = 0$$

(ii) $\sup_N \|S_N\|_{op} = M < \infty$ $\quad (S_N : L^p(\mathbb{T}) \to L^p(\mathbb{T}))$.

Proof. (i) $\Rightarrow$ (ii) Assume $\lim_{N\to\infty} \|S_N f - f\|_p = 0$. Then

$$\big|\, \|S_N(f)\|_p - \|f\|_p \big| \leq \|S_n(f) - f\|_p$$

(by the reverse triangle inequality) implies

$$\lim_{N\to\infty} \|S_N(f)\|_p = \|f\|_p < \infty.$$

Since $\| S_N(f)\|_p$ converges (as $N\to\infty$) to a finite number ($\|f\|_p$), for each f, $\sup_{N\in\mathbb{N}} \| S_N(f)\| \le M_f < \infty$. By the uniform boundedness principle (4.1.6) we conclude

$\sup_N \| S_N\|_{op} = M < \infty$ as wanted.

(ii) $\Rightarrow$ (i) Write

$$\| S_N(f) - f\|_p = \| S_N(f-\sigma_N(f)) + S_N(\sigma_N f) - f)\|_p$$

Since $\sigma_N(f)$ is a finite linear combination of $\{e_n\}_{|n|\in\mathbb{N}}$ (see the proof of Corollary 8.3) it follows that $S_N(\sigma_N f) = \sigma_N(f)$. Thus

$$\begin{aligned}\|S_N(f) - f\|_p &\le \|S_N(f-\sigma_N(f))\|_p + \|\sigma_N(f) - f\|_p \\ &\le \| S_N\|_{op} \|f - \sigma_N(f)\|_p + \|\sigma_N(f) - f\|_p \\ &= (\|S_N\|_{op} + 1)\|f - \sigma_N(f)\|_p \le (M+1)\|f - \sigma_N(f)\|_p.\end{aligned}$$

By Theorem 8.2, given $\varepsilon > 0$, there exists $N_0 = N(\varepsilon)\in\mathbb{N}$ such that $\| f - \sigma_N(f)\|_p < \frac{\varepsilon}{M+1}$ for all $N \ge N_0$. Thus

$$\|S_N f - f\|_p \le (M+1)\frac{\varepsilon}{M+1} = \varepsilon$$

QED.

<u>Corollary 8.5</u> Fourier series do not converge on $L^1(\mathbb{T})$, that is, there exists $f\in L^1(\mathbb{T})$ such that $\lim_{N\to\infty} \|S_N(f) - f\|_1 \neq 0$.

<u>Proof</u> By Proposition 8.4 it is enough to show that

$$\sup_N \|S_N\|_{op} = \infty$$

where

$$\|S_N\|_{op} = \sup_{\|f\|_1 = 1} \|S_N(f)\|_1$$

Recall that $F_M \in L^1(\mathbb{T})$ (Fejér kernel) and $\|F_M\|_1 = 1$. We have

$$\|S_N\|_{op} = \sup_{\|f\|_1=1} \|S_N(f)\|_1 \geq \|S_N(F_M)\|_1$$

$$= \|D_N * F_M\|_1 \longrightarrow \|D_N\|_1 \quad \text{as } N \to \infty$$

Since $\|D_N\|_1 \approx \ln N$ we conclude using Proposition 8.4. QED

Exercise 8.D Show that the Fourier series does not converge in $(C(\mathbb{T}), \|\ \|_\infty)$, that is, there exists $f \in C(\mathbb{T})$ such that $\lim_{N\to\infty} \|S_N(f) - f\|_\infty \neq 0$

By Prop 8.4 it is enough to show that

$$\sup_N \|S_N\|_{op} = \infty$$

where

$$\|S_N\|_{op} = \sup_{\|f\|_\infty=1} \|S_N(f)\|_\infty$$

We have

$$\|S_N\|_{op} = \sup_{\|f\|_\infty=1} \|D_N * f\|_\infty \geq \sup_{\|f\|_\infty=1} |D_N * f(0)|$$

$$= \sup_{\|f\|_\infty=1} \left|\int_{\mathbb{T}} D_N(-y) f(y)\, dy\right|$$

Take $f_0 = \operatorname{sign} D_N(-y)$ (observe that $\|f_0\|_\infty = 1$. Then

$$\|S_N\|_{op} \geq \int_{\mathbb{T}} |D_N(-y)|\, dy = \|D_N\|_1 \to \infty \quad \text{as } N \to \infty.$$

Warning We have proved the existence of continuous functions whose Fourier series do not converge uniformly to the function. This is not an explicit examples. For explicit examples see the book by T.W. Körner, Fourier Analysis, 2nd Edition, Cambridge U. Press, 1989).

For convergence of Fourier series in $L^p([0,1])$, $1<p<\infty$, $(p\neq 2)$, one needs to work more.

For a "trigonometric" function, that is a finite linear combination of exponentials of the form $f(x)=\sum_{k\in F}\hat{f}(k)e^{2\pi i k x}$ where $F\subset\mathbb{Z}$ is finite, the conjugate function of f is

$$\tilde{f}(x)=-i\sum_{k\in F}\operatorname{sign}(k)\hat{f}(k)e^{2\pi i k x}$$

Proposition 8.6. Let $1<p<\infty$. The following are equivalent

(a) $\lim_{N\to\infty}\|S_N(f)-f\|_p=0$, $f\in L^p(\mathbb{T})$

(b) There exists $0<C_p<\infty$ such that $\|\tilde{f}\|_p\leq C_p\|f\|_p$ for all "trigonometric" polynomials f.

The key result is now the following

Theorem 8.7 Let $1<p<\infty$. There exists a constant $0<A_p<\infty$ shuch that for all $f\in L^\infty(\mathbb{T})$ we have

$$\|\tilde{f}\|_p\leq A_p\|f\|_p.$$

Since the trigonometric polynomials are all bounded functions, Theorem 8.7 gives item (b) of Prop. 8.6. Therefore (a) holds. This shows that the exponential sequence

$$\mathcal{E}=\{e_0, e_1, e_{-1}, e_2, e_{-2}, \dots\}$$

with this ordering, is a basis for $(L^p(\mathbb{T}), \|\ \|_p)$ when $1<p<\infty$.

Note: This result can be found in Katznelson' book (page 59).

Although the exponentials $\mathcal{E}$ form a basis for $L^p(\mathbb{T})$, $1<p<\infty$, in the order given, it is only unconditional if $p=2$.

One way to see that $\mathcal{E}$ is not unconditional in $L^p(\mathbb{T})$ if $p\neq 2$ is to borrow a result from Zygmund's book (Trigonometric series, Cambridge U. Press, 1959): there exists a function $f\in L^p(\mathbb{T})$, $p\neq 2$, $1<p<\infty$, such that the function

$$\sum_{k\in\mathbb{Z}} \varepsilon_k c_k e^{2\pi i k x}, \quad c_k=\int_0^1 f(x)e^{-2\pi i k x}\,dx$$

is not in $L^p(\mathbb{T})$, $p\neq 2$, for almost all choices of signs $\varepsilon=\{\varepsilon_n\}_{n=1}^{\infty}$ with $\varepsilon_n=\pm 1$. Then, the non unconditionality of $\mathcal{E}$ in $L^p(\mathbb{T})$, $1<p<\infty$, $p\neq 2$, follows from Theorem 7.8.

Note: The first publications of results of this type can be found in R.E.A.C. Paley and A Zygmund, A remarkable system of orthogonal functions, Proc. of the Cambridge Phil. Soc 26 (1930), 337-357 and 28 (1932), 190-205.

Another way to see this is to use the following result

Thm (W. Orlicz) If the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally in $L^p([0,1))$ with $1\le p\le 2$, then $\sum_{n=1}^{\infty}\|x_n\|^2<\infty$

Proof. See P. Wojtaszczyk book, page 60

P. Wojtaszczyk, Banach Spaces for Analysts, Cambridge U. Press, 1991

Proposition 8.8 The exponential system $\mathcal{E}=\{e^{2\pi i n x}\}_{n=-\infty}^{\infty}$ is an unconditional basis in $L^p([0,1))$, $1<p<\infty$, only if $p=2$

P/ We know that $\mathcal{E}$ is an unconditional basis of $L^2([0,1))$ because it is an orthonormal basis (Exercise 7.D).

Consider $1\le p<2$. Suppose that the series $\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ represents a function $f\in L^p([0,1))$ with unconditional convergence. By Orlicz's Thm,

$$\sum_{n=-\infty}^{\infty} \|a_n e^{2\pi i n \cdot}\|_p^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$$

This implies that $f\in L^2([0,1))$. But we know that for $1\le p<2$, $L^2([0,1)) \subsetneq L^p([0,1))$. (For example, $f(x)=x^{-1/2}\in L^p([0,1))$, but $f\notin L^2([0,1))$)

For $2<p<\infty$, let p' such that $\frac{1}{p'}+\frac{1}{p}=1$. Then $1<p'<2$

Assume that $\mathcal{E}$ where unconditional in $L^p([0,1))$, let

$$f_N(x)=\sum_{n=1}^{N} a_n e^{2\pi i n x}, \quad g_N(x)=\sum_{n=1}^{N} \varepsilon_n a_n e^{2\pi i n x}$$

for $a_1,\dots,a_N\in\mathbb{C}$, $\varepsilon_n=\pm1$. Then

$$\|g_N\|_{p'} = \sup_{\|g_N'\|_p=1} \int_0^1 g_N(x)\overline{g_N'(x)}\,dx$$

Remark This supremum should be taken over all $g'\in L_p([0,1])$, $\|g'\|_p=1$; but since $\mathcal{E}$ is an orthonormal systems, coefficients of g' with frequency greater than N, give integral zero

Write $\overline{g_N'(x)}=\sum_{n=1}^{N} b_n e^{2\pi i n x}$ for some $\{b_n\}_{n=1}^{N}\subset\mathbb{C}$. Let $f_N'(x)=\sum_{n=1}^{N}\varepsilon_n b_n e^{2\pi i n x}$.

Then

$$\int_0^1 g_N(x)\overline{g_N'(x)}\,dx = \int_0^1 \Big(\sum_{n=1}^{N}\varepsilon_n a_n e^{2\pi i n x}\Big)\Big(\sum_{n=1}^{N} b_n e^{2\pi i n x}\Big)dx$$

$$= \int_0^1 \Big(\sum_{n=1}^{N} a_n e^{2\pi i n x}\Big)\Big(\sum_{n=1}^{N}\varepsilon_n b_n e^{2\pi i n x}\Big)dx = \int_0^1 f_N(x) f_n'(x)\,dx$$

$$\overset{\text{Hölder's}}{\le} \|f_N\|_{p'}\|f_N'\|_p .$$

Since we are assuming (contrary to what we want to prove) that $\mathcal{E}$ is unconditional in $L^p([0,1])$, $\|f_N'\|_p \leq k\|g_N'\|_p$ (by Exercise 7.A).

Then

$$\|g_N\|_{p'} \leq \|f_N\|_{p'}\, k\|g_N'\|_p = k\|f_N\|_{p'},$$

and thus will show that $\mathcal{E}$ is unconditional in $L^{p'}([0,1])$, $1<p'<2$, which we have proved to be not true.

Therefore, the exponential basis is not unconditional in $L^p([0,1])$ if $2<p<\infty$.