7. UNCONDITIONAL BASES

"The existence of Schauder basis in a Banach space does not give very much information on the structure of the space. If one wants to study in more detail the structure of a Banach space by using bases one is led to consider bases with various special properties. ..... Undoubtedly, the most useful and widely studied special class of bases is that of unconditional bases."

Lindenstrauss, Tzafriri, Classical Banach Spaces I, Springer, (1977)

Let $\mathcal{F}(\mathbb{N})$ denote the collection of all finite subsets $\mathcal{N}$ of $\mathbb{N}$.

Def 7.1. Let $(V, \|\ \|)$ be a normed vector space and $\{y_n\}_{n\in\mathbb{N}} \subset V$. We say that $\sum\limits_{n\in\mathbb{N}} y_n$ converges unconditionally to $y \in V$, and write

$$\lim_{\mathcal{N}\in\mathcal{F}(\mathbb{N})} \sum_{n\in\mathcal{N}} y_n \overset{UC}{=} y \qquad (7.1)$$

if for every $\varepsilon > 0$, there exists an $\mathcal{N} = \mathcal{N}(\varepsilon) \in \mathcal{F}$ such that

$$\|y - \sum_{n\in\mathcal{N}'} y_n\| \leq \varepsilon \quad \text{for all } \mathcal{N}' \in \mathcal{F}(\mathbb{N}) \text{ with } \mathcal{N}' \supset \mathcal{N}$$

Def 7.2. A basis $\mathcal{B} = \{x_n\}_{n\in\mathbb{N}}$ of a Banach space $(\mathbb{B}, \|\ \|)$ is said to be an unconditional basis if in the unique representation $x = \sum\limits_{n\in\mathbb{N}} \alpha_j x_j$, $x \in \mathbb{B}$, the series converges unconditionally

The following lemma shows, as expected, that if a series converges unconditionally to y, any permutation of the elements of the series produces a new series converging in norm to the same element y.

Lemma 7.3 Let $(\mathbb{B}, \|\ \|)$ be a Banach space and $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{B}$.
The following are equivalent

1. $\sum_{n\in\mathbb{N}} y_n$ converges unconditionally to $y \in \mathbb{B}$
2. $\sum_{n=1}^{\infty} y_{\sigma(n)} = y$ (convergence in $\|\ \|$) for all permutations σ of $\mathbb{N}$.

P/ 1. $\Rightarrow$ 2. Suppose that $\sum_{n\in\mathbb{N}} y_n$ converges unconditionally to $y \in \mathbb{B}$ and let σ be a permutation of $\mathbb{N}$. Given $\varepsilon > 0$, choose $\mathcal{N} = \mathcal{N}(\varepsilon) \in \mathcal{F}(\mathbb{N})$ such that

$$\|y - \sum_{n\in\mathcal{N}'} y_n\| \leq \varepsilon \quad \text{for all } \mathcal{N}' \in \mathcal{F}(\mathbb{N}) \text{ with } \mathcal{N}' \supset \mathcal{N}.$$

Let n_0 be defined by $n_0 := \sup\{n \in \mathbb{N} : \sigma(n) \in \mathcal{N}(\varepsilon)\}$. If $n \geq n_0$ we have

$$\|\sum_{j=1}^{n} y_{\sigma(j)} - y\| = \|\sum_{k\in\{\sigma(j)\,:\, j\leq n\}} y_k - y\| \leq \varepsilon$$

since $\{\sigma(j) : j \leq n\} \supseteq \mathcal{N}(\varepsilon)$.

2. $\Rightarrow$ 1. Assume now that $\sum_{n=1}^{\infty} y_{\sigma(n)} = y$ (in $\|\ \|$) for all permutation σ of $\mathbb{N}$ and suppose that

$$\lim_{\mathcal{N}\in\mathcal{F}(\mathbb{N})} \sum_{n\in\mathcal{N}} y_n \overset{UC}{\neq} y.$$

Then, we can find an $\varepsilon_0 > 0$ such that for each $\mathcal{N} \in \mathcal{F}(\mathbb{N})$ there exists $\mathcal{N}' \supset \mathcal{N}$, $\mathcal{N}' \in \mathcal{F}(\mathbb{N})$, and

$$\|\sum_{n\in\mathcal{N}'} y_n - y\| > \varepsilon_0 .$$

Let us start with $\mathcal{N}_0 = \{1\}$ and let $\mathcal{N}_0' \in \mathcal{F}(\mathbb{N})$ with $\mathcal{N}_0' \supset \mathcal{N}_0$ be such that

$$\|\sum_{n\in\mathcal{N}_0'} y_n - y\| > \varepsilon_0 .$$

Let $n_1 = \sup\{k \in \mathbb{N} : k \in \mathcal{N}_0'\}$ and put $\mathcal{N}_1 = \{1, 2, \ldots, n_1 + 1\}$. We can find $\mathcal{N}_1' \in \mathcal{F}(\mathbb{N})$ with $\mathcal{N}_1' \supset \mathcal{N}_1$ such that

$$\Big\| \sum_{n \in \mathcal{N}_1'} y_n - y \Big\| > \varepsilon_0 .$$

Continuing in this way we obtain a sequence of elements of $\mathcal{F}(\mathbb{N})$

$$\mathcal{N}_0 \subset \mathcal{N}_0' \subset \mathcal{N}_1 \subset \mathcal{N}_1' \subset \dots \subset \mathcal{N}_k \subset \mathcal{N}_k' \subset \dots$$

Such that

$$\Big\| \sum_{n \in \mathcal{N}_k'} y_n - y \Big\| > \varepsilon_0 \quad \text{for all } k = 0, 1, 2, 3, \dots$$

Observe that

$$\mathcal{N}_0,\ \mathcal{N}_0' \setminus \mathcal{N}_0,\ \mathcal{N}_1 \setminus \mathcal{N}_0', \dots,\ \mathcal{N}_k' \setminus \mathcal{N}_k,\ \mathcal{N}_{k+1} \setminus \mathcal{N}_k' \dots$$

is a disjoint union with equals $\mathbb{N}$. Reorder $\mathbb{N}$ by letting the elements of the n^{th} set of the this union precede the elements of the $(n+1)^{th}$ set, $n = 0, 1, 2, \dots$ This defines a permutation σ of $\mathbb{N}$ satisfying

$$\sum_{n=1}^{\operatorname{card}(\mathcal{N}_k')} y_{\sigma(n)} = \sum_{n \in \mathcal{N}_k'} y_n$$

Since $\| \sum_{n \in \mathcal{N}_k'} y_n - y \| > \varepsilon_0$, the series $\sum_{n=1}^{\infty} y_{\sigma(n)}$ cannot converge to y in the norm of $\mathbb{B}$. QED

<u>Lemma 7.4</u> Let $(\mathbb{B}, \| \ \|)$ be a Banach space and $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{B}$. Suppose $\sum_{n \in \mathbb{N}} x_n$ converges unconditionally to $x \in \mathbb{B}$. Then,

$$\sum_{n=1}^{\infty} |f(x_n)|$$

converges <u>uniformly</u> for all $f \in \mathbb{B}_1^* = \{ f \in \mathbb{B}^* : \|f\|_* \le 1 \}$ (the closed unit ball of $\mathbb{B}^*$). That is, given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\boxed{\sum_{j=n+1}^{n+k} |f(x_j)| \le \varepsilon \quad \text{whenever } n \ge N \text{ and } k \in \mathbb{N}, \text{ and all } f \in \mathbb{B}_1^*}$$

Proof. Suppose $\lim\limits_{\mathcal{N} \in \mathcal{F}(\mathbb{N})} \sum\limits_{n \in \mathcal{N}} x_n \overset{UC}{=} x$ and $\varepsilon > 0$. Let $\mathcal{N} = \mathcal{N}(\varepsilon) \in \mathcal{F}(\mathbb{N})$ be such that

$$\Big\| x - \sum_{n \in \mathcal{N}'} x_n \Big\| \le \frac{\varepsilon}{8} \text{ for all } \mathcal{N}' \in \mathcal{F}(\mathbb{N}),\ \mathcal{N}' \supset \mathcal{N}. \qquad (7.2)$$

Let $N = N(\varepsilon) := \max\{n : n \in \mathcal{N}(\varepsilon)\}$. For $n \ge N$, $k \ge 1$ and $f \in \mathbb{B}_1^*$ we define

$$\mathcal{N}_1(f) = \{j : n+1 \le j \le n+\ell,\ \mathrm{Re}\, f(x_j) \ge 0\} \text{ and}$$

$$\mathcal{N}_2(f) = \{i : n+1 \le j \le n+\ell,\ \mathrm{Re}\, f(x_j) < 0\}$$

Then

$$\sum_{j=n+1}^{n+k} |\mathrm{Re}\, f(x_j)| = \Big| \mathrm{Re}\, f\Big(\sum_{j \in \mathcal{N}_1(f)} x_j\Big)\Big| + \Big|\mathrm{Re}\, f\Big(\sum_{j \in \mathcal{N}_2(f)} x_j\Big)\Big|$$

$$\le \|f\|_* \Big(\Big\|\sum_{j \in \mathcal{N}_1(f)} x_j\Big\| + \Big\|\sum_{j \in \mathcal{N}_2(f)} x_j\Big\|\Big) \overset{\|f\|_* \le 1}{\le}$$

$$\le \Big\|\sum_{j \in \mathcal{N}_1(f)} x_j\Big\| + \Big\|\sum_{j \in \mathcal{N}_2(f)} x_j\Big\| =: \mathrm{I} + \mathrm{II}.$$

For I we have

$$\mathrm{I} = \Big\| x - \sum_{j \in \mathcal{N}(\varepsilon)} x_j - \sum_{j \in \mathcal{N}_1(f)} x_j - x + \sum_{j \in \mathcal{N}(\varepsilon)} x_j \Big\|$$

$$\le \Big\| x - \sum_{j \in \mathcal{N}(\varepsilon) \cup \mathcal{N}_1(f)} x_j \Big\| + \Big\| x - \sum_{j \in \mathcal{N}(\varepsilon)} x_j \Big\| \le \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}$$

by (7.2) because $\mathcal{N}(\varepsilon) \cup \mathcal{N}_1(f)$ and $\mathcal{N}(\varepsilon)$ contain $\mathcal{N} = \mathcal{N}(\varepsilon)$.

Similarly, we obtain $\mathrm{II} \le \varepsilon/4$. We can do the same thing with $\mathrm{Im}\, f(x_j)$ and, hence,

$$\sum_{j=n+1}^{n+k} |f(x_j)| \le \sum_{j=n+1}^{n+k} |\mathrm{Re}\, f(x_n)| + \sum_{j=n+1}^{n+k} |\mathrm{Img}\, f(x_n)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \quad \text{QED.}$$

Although lemma 7.3 gives us a good intuitive idea of what unconditional convergence is, it is of less use when we need to prove that a basis is unconditional. The next results will give characterizations of unconditional convergence that are more suitable to decide if a series is unconditional convergent

Theorem 7.5 Let $(\mathbb{B}, \|\ \|)$ be a Banach space and $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{B}$. The following are equivalent

(i) The series $\sum_{n\in\mathbb{N}} x_n$ converges unconditionally

(ii) The series $\sum_{n=1}^{\infty} \beta_n x_n$ converges for every sequence $\{\beta_n\}_{n=1}^{\infty}$ such that $|\beta_n| \le 1$ for all $n \in \mathbb{N}$

(iii) The series $\sum_{j=1}^{\infty} x_{n_j}$ converges for every increasing subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$

(iv) The series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges for every sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\varepsilon_n = \pm 1$.

P/ (i) $\Rightarrow$ (ii) Let $\varepsilon > 0$. By lemma 7.4 we can choose an $N = N(\varepsilon)$ such that

$$\sum_{j=n+1}^{n+k} |f(x_j)| \le \varepsilon \quad \text{for all } f \in \mathbb{B}_1^*, \text{ when } n \ge N \text{ and } k \ge 1. \quad (7.4)$$

then, by Proposition 5.6 (a corollary of the Hahn-Banach theorem)

$$\Big\| \sum_{j=n+1}^{n+k} \beta_j x_j \Big\| = \sup_{f\in\mathbb{B}_1^*} \Big| f\Big(\sum_{j=n+1}^{n+k} \beta_j x_j \Big) \Big| = \sup_{f\in\mathbb{B}_1^*} \Big| \sum_{j=n+1}^{n+k} \beta_j f(x_j) \Big| .$$

Since $|\beta_j| \leq 1$, using (7.4) we obtain

$$\Big\| \sum_{j=k+1}^{n+k} \beta_j x_j \Big\| \leq \sup_{f \in \mathbb{B}_1^*} \sum_{j=n+1}^{n+k} |f(x_j)| \leq \varepsilon .$$

This shows that the partial sums $\sum_{j=1}^{n} \beta_j x_j$ form a Cauchy sequence in $\mathbb{B}$. Since $(\mathbb{B}, \| \ \|)$ is complete, the series $\sum_{j=1}^{\infty} \beta_j x_j$ converges to an element of $\mathbb{B}$.

(ii) $\Rightarrow$ (iii) Given an increasing sub. sequence $n_1 < n_2 < n_3 < \dots$ of integers, choose

$$\beta_k = \begin{cases} 1 & \text{if } k = n_j \text{ for some } j \\ 0 & \text{otherwise} \end{cases}.$$

and apply (ii) to show that the series $\sum_{j=1}^{\infty} x_{n_j}$ converges in $\mathbb{B}$.

(iii) $\Leftrightarrow$ (iv) Given a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ with $\varepsilon_j = \pm 1$, choose

$$\beta_j = \begin{cases} 1 & \text{if } \varepsilon_j = 1 \\ 0 & \text{if } \varepsilon_j = -1 \end{cases}$$

Then, for every $n \in \mathbb{N}$

$$\sum_{j=1}^{n} \beta_j x_j = \frac{1}{2}\Big(\sum_{j=1}^{n} \varepsilon_j x_j + \sum_{j=1}^{n} x_j \Big) \qquad (7.5)$$

If (iii) holds, the series $\sum_{j=1}^{\infty} \beta_j x_j$ converges in $\mathbb{B}$ and also $\sum_{j=1}^{\infty} x_j$ converges in $\mathbb{B}$. By (7.5), $\sum_{j=1}^{n} \varepsilon_j x_j$ converges in $\mathbb{B}$. This shows that (iii) $\Rightarrow$ (iv)

Take now an increasing subsequence $n_1 < n_2 < n_3 < \dots$ of positive integers. Define

$$\varepsilon_k = \begin{cases} 1 & \text{if } k = n_j \text{ for some } j \\ -1 & \text{if otherwise} \end{cases}$$

Then $$\sum_{j=1}^{n} x_{n_j} = \frac{1}{2}\Big(\sum_{k=1}^{n} \varepsilon_k x_k + \sum_{k=1}^{n} x_k \Big) \qquad (7.6)$$

If (iv) holds, both series on the right hand side of (7.6) converges in $\mathbb{B}$. Therefore, the one on the left hand side also converges in $\mathbb{B}$.

(iii) $\Rightarrow$ (i) Assume that $\sum_{j=1}^{\infty} x_{n_j}$ is convergent for all strictly increasing subsequence of natural numbers

$$n_1 < n_2 < \dots < n_r < \dots$$

In particular $\sum_{j=1}^{\infty} x_n$ is a well-defined element of $\mathbb{B}$ that converges to an element $x \in \mathbb{B}$. Suppose that $\sum_{n \in \mathbb{N}} x_n$ is not unconditionally convergent. In particular, it does not converge unconditionally to x. Thus, there exists $\varepsilon_0 > 0$ such that for all $\mathcal{N} \in \mathcal{F}(\mathbb{N})$ we can find $\mathcal{N}' \in \mathcal{F}(\mathbb{N})$ with $\mathcal{N} \subset \mathcal{N}'$ and

$$\Big\| \sum_{n \in \mathcal{N}'} x_n - x \Big\| > \varepsilon_0$$

Let $\mathcal{N}_0 = \{1\}$. Then, there exists $\mathcal{N}_0' \in \mathcal{F}(\mathbb{N})$ such that $\mathcal{N}_0 \subset \mathcal{N}_0'$ and

$$\Big\| \sum_{n \in \mathcal{N}_0'} x_n - x \Big\| > \varepsilon_0$$

Let $n_1 = \max\{n : n \in \mathcal{N}_0'\}$ and $\mathcal{N}_1 = \{1, 2, \dots, n_1 + 1\}$. Then, there exits $\mathcal{N}_1' \in \mathcal{F}(\mathbb{N})$ such that $\mathcal{N}_1 \subset \mathcal{N}_1'$ and

$$\Big\| \sum_{n \in \mathcal{N}_1'} x_n - x \Big\| > \varepsilon_0$$

Continuing this process we obtain an increasing sequence of finite subsets of $\mathbb{N}$,

$$\mathcal{N}_0 \subset \mathcal{N}_0' \subset \mathcal{N}_1 \subset \mathcal{N}_1' \subset \dots \subset \mathcal{N}_{k-1}' \subset \mathcal{N}_k \subset \mathcal{N}_k' \subset \dots$$

such that $n_k = \max\{n : n \in \mathcal{N}_{k-1}'\}$, $\mathcal{N}_k = \{1, 2, \dots, n_k + 1\}$, and

$$\Big\| \sum_{n \in \mathcal{N}_k'} x_n - x \Big\| > \varepsilon_0 \qquad \forall k = 1, 2, \dots \qquad (7.7)$$

Let $\mathcal{D}_k = \mathcal{N}_k' - \mathcal{N}_k$, $k=1,2,\ldots$ Then $\mathcal{D}_k \subset \{n_1+2, \ldots, n_{k+1}+1\}$ where $n_k+2 \le n_{k+1}+1$. Then, the collection $\mathcal{D}_k$ is strictly disjoint and

$$\max\{n : n \in \mathcal{D}_k\} < \min\{n : n \in \mathcal{D}_{k+1}\}.$$

The union of $\mathcal{D}_k'$ in their natural order is an increasing subsequences $\{n_j\}_{j=1}^{\infty}$ of natural numbers. From (7.7) we have

$$0 < \varepsilon_0 \le \Big\| \Big(\sum_{n \in \mathcal{N}_k} x_n - x\Big) + \sum_{n \in \mathcal{D}_k} x_n \Big\|$$

$$\le \Big\| \sum_{n \in \mathcal{N}_k} x_n - x \Big\| + \Big\| \sum_{n \in \mathcal{D}_k} x_n \Big\| =$$

$$= \Big\| \sum_{n=1}^{n_k+1} x_n - x \Big\| + \Big\| \sum_{n \in \mathcal{D}_k} x_n \Big\|. \qquad (7.8)$$

The sum $\sum_{n \in \mathcal{D}_k} x_n$ is the difference of two partial sums of the convergent series $\sum_{j=1}^{\infty} x_{n_j}$. Therefore, $\lim_{k \to \infty} \sum_{n \in \mathcal{D}_k} x_n = 0$. Hence, we can find k_0 large enough so that $\| \sum_{n \in \mathcal{D}_k} x_n \| < \frac{\varepsilon_0}{4}$, for all $k \ge k_0 \ldots$

Also, since $\sum_{n=1}^{\infty} x_n = x$, there exists $k_1 \in \mathbb{N}$ such that $\| \sum_{n=1}^{n_k+1} x_n - x \| \le \frac{\varepsilon_0}{4}$ if $k \ge k_1$. For $k \ge \max\{k_0, k_1\}$ in (7.8) we obtain $0 < \varepsilon_0 < \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{2}$, which is a contradiction. QED

Recall that a basis $\mathcal{B} = \{x_n\}_{n=1}^{\infty}$ of a Banach space $(\mathbb{B}, \|\ \|)$ is an unconditional basis for $\mathbb{B}$ if in the <u>unique</u> representation

$$x = \sum_{n=1}^{\infty} x_n^*(x)\, x_n \qquad , x \in \mathbb{B},$$

the series converges unconditionally.

Theorem 7.6 Let $\mathcal{B}=\{x_n\}_{n=1}^{\infty}$ be a basis for a Banach space $(\mathbb{B}, \|\ \|)$. The following are equivalent:

(i) $\mathcal{B}$ is an unconditional basis of $(\mathbb{B}, \|\ \|)$

(ii) For every permutation $\sigma \in \mathbb{N}$, the set $\mathcal{B}^{\sigma} := \{x_{\sigma(n)}\}_{n=1}^{\infty}$ is a basis for $(\mathbb{B}, \|\ \|)$

(iii) For every $x \in \mathbb{B}$ with unique representation $x = \sum_{n=1}^{\infty} x_n^*(x) x_n$ and every sequence $\{\beta_n\}_{n=1}^{\infty}$ such that $|\beta_n| \leq 1$, the series $\sum_{n=1}^{\infty} \beta_n x_n^*(x) x_n$ converges in $(\mathbb{B}, \|\ \|)$.

P/ It is a Corollary of lemma 7.3 and Theorem 7.5.

Remarks Conditions (iii) and (iv) in Theorem 7.5 give two more equivalent conditions

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Let $\mathcal{B}=\{x_n\}_{n=1}^{\infty}$ be an unconditional basis of a Banach space $(\mathbb{B}, \|\ \|)$. By definition, for every $x \in \mathbb{B}$, the series $\sum_{n=1}^{\infty} x_n^*(x) x_n$ converges unconditionally. By part (iii) of Theorem 7.6, for every $\beta = \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{F}$ $(|\beta_n| \leq 1)$ the series $\sum_{n=1}^{\infty} \beta_n x_n^*(x) x_n$ converges in $(\mathbb{B}, \|\ \|)$. Therefore, the operator

$$S_{\beta}(x) = \sum_{n=1}^{\infty} \beta_n x_n^*(x) x_n, \quad x \in \mathbb{B} \qquad (7.9)$$

maps $\mathbb{B}$ into $\mathbb{B}$. It is linear since x_n^* is linear.

We shall prove that S_{β} is continuous. To prove this we shall use the closed graph theorem.

Suppose $\{x^{(k)}\}_{k=1}^{\infty}$ converges to x in $(\mathbb{B}, \|\ \|)$ and $\{S_{\beta}(x^{(k)})\}_{k=1}^{\infty}$ converges to y in $(\mathbb{B}, \|\ \|)$. We need to prove that $S_\beta(x) = y$. Write

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n , \quad x^{(k)} = \sum_{n=1}^{\infty} x_n^*(x^{(k)}) x_n , \quad y = \sum_{n=1}^{\infty} x_n^*(y) x_n .$$

Since S_β is linear,

$$S_\beta(x^{(k)}) = \sum_{n=1}^{\infty} \beta_n x_n^*(x^{(k)}) x_n . \qquad (7.10)$$

Since the coefficient functionals x_n^* are continuous $\beta_n x_n^*(x^{(k)}) \longrightarrow \beta_n x_n^*(x)$ as $k \to \infty$. Also

$$\beta_n x_n^*(x^{(k)}) = x_n^*(\beta_n x^{(k)}) \overset{(7.10)}{=} x_n^*(S_\beta(x^{(k)})) \xrightarrow{k \uparrow \infty} x_n^*(y).$$

Hence $\beta_n x_n^*(x) = x_n^*(y)$ which shows that

$$S_\beta(x) = \sum_{n=1}^{\infty} \beta_n x_n^*(x) x_n = \sum_{n=1}^{\infty} x_n^*(y) x_n = y .$$

This proves that $S_\beta : \mathbb{B} \mapsto \mathbb{B}$ is bounded (Theorem 4.2). But the bound may vary with β. We shall show that $\|S_\beta\|_{op}$ has a bound independent of β

<u>Lemma 7.7</u> Let $\beta = \{x_n\}_{n=1}^{\infty}$ be an unconditional basis for a Banach space $(\mathbb{B}, \|\ \|)$. Then, there exists a constant $C < \infty$ such that for all $x \in \mathbb{B}$

$$\|S_\beta(x)\| \leq C \|x\| ,$$

for all sequences $\beta = \{\beta_n\}_{n=1}^{\infty}$ such that $|\beta_n| \leq 1$, where S_β is defined in (7.9)

<u>Proof</u>. We have shown that S_β is a bounded linear operator on $\mathbb{B}$. Let $x = \sum_{n=1}^{\infty} x_n^*(x) x_n \in \mathbb{B}$ and $f \in \mathbb{B}_1^* = \{ f \in \mathbb{B}^* : \|f\|_* \leq 1\}$.
By Lemma 7.4 with $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |x_n^*(x) f(x_n)| \leq 1 \quad \text{for all } f \in \mathbb{B}_1^* .$$

Thus,

$$\sum_{n=1}^{\infty} |x_n^*(x)\, f(x_n)| \leq \sum_{n=1}^{N} |x_n^*(x)\, f(x_n)| + 1 := c(x) + 1$$

For each $f \in \mathbb{B}_1^*$ we can, define $T_f : \mathbb{B} \to \ell^1(\mathbb{N})$ by

$$T_f(x) = \{ x_n^*(x) f(x_n) \}_{n \geq 1}^{\infty} \quad , x \in \mathbb{B}$$

Each T_f is linear. because x_n^* is linear. Also, if $\|x\| = 1$

$$\|T_f(x)\|_{\ell^1} = \sum_{n \geq 1}^{\infty} |x_n^*(x) f(x_n)| \leq \sum_{n \geq 1}^{N} |x_n^*(x) f(x_n)| + \|x\|$$

$$\leq \|x\| \, \|f\|_* \sum_{n \geq 1}^{N} \|x_n^*\|_* \|x_n\| + \|x\| \overset{\text{Prop 6.13}}{\leq}$$

$$\leq \|x\| (MN + 1)$$

This shows that T_f is bounded and that the set

$$\{ \|T_f(x)\|_{\ell^1} : f \in \mathbb{B}_1^* \}$$

is bounded by $c(x) + 1$. By the Banach - Steinhauss Theorem (or uniform boundedness principle), there exists $C < \infty$ such that $\|T_f\|_{op} \leq C$ for all $f \in \mathbb{B}_1^*$. Hence

$$\|T_f(x)\|_{\ell^1} \leq C \|x\| \quad \text{for all } f \in \mathbb{B}_1^*.$$

Therefore,

$$|f(S_\beta(x))| = \Big| f\Big(\sum_{n=1}^{\infty} \beta_n x_n^*(x) x_n\Big)\Big| = \Big|\sum_{n=1}^{\infty} \beta_n x_n^*(x) f(x_n)\Big|$$

$$\leq \sum_{n \geq 1}^{\infty} |x_n^*(x) f(x_n)| = \|T_f(x)\|_{\ell^1} \leq C \|x\|$$

for all $x \in \mathbb{B}$ and all $f \in \mathbb{B}_1^*$. Finally, by Prop 4.6

$$\|S_\beta(x)\|_{\mathbb{B}} = \sup_{f \in \mathbb{B}_1^*} |f(S_\beta(x))| \leq C \|x\|. \qquad \text{QED}$$

The converse of lemma 7.7 is also true. In fact, we have the following stronger result.

Lemma 7.8. Let $\mathcal{B}=\{x_n\}_{n\geq 1}^{\infty}$ be a basis in a Banach space $(\mathbb{B}, \|\ \|)$. Suppose that there exists a constant $C<\infty$ such that the operator S_β defined in (7.9) satisfies $\|S_\beta(x)\|\leq C\|x\|$ for all $x\in\mathbb{B}$ and finitely non-zero $\beta=\{\beta_n\}_{n\geq 1}^{\infty}$ with $\beta_1=1$ or 0. Then the basis $\mathcal{B}$ is unconditional.

Proof. Let $x=\sum_{n\geq 1}^{\infty} x_n^*(x)x_n\in\mathbb{B}$ and σ any permutation of $\mathbb{N}$.
By Lemma 7.3 it is enough to show that the series

$$\sum_{n\geq 1}^{\infty} x_{\sigma(n)}^*(x)\,x_{\sigma(n)}$$

converges to x in $\mathbb{B}$. The convergence of $\sum_{n\geq 1}^{\infty} x_n^*(x)x_n$ to x in $\mathbb{B}$ implies that, for given $\varepsilon>0$, there exists an $N=N(\varepsilon)\in\mathbb{N}$ such that

$$\|x-\sum_{j\geq 1}^{n} x_j^*(x)x_n\|\leq\frac{\varepsilon}{2C+1}\quad\text{for all } n\geq N=N(\varepsilon). \tag{7.11}$$

Let $M=M(\varepsilon,\sigma)\in\mathbb{N}$ such that $\{1,2,\dots,N\}\subseteq\{\sigma(1),\sigma(2),\dots,\sigma(M)\}$.
If $m\geq M=M(\varepsilon,\sigma)$

$$\|x-\sum_{j\geq 1}^{m} x_{\sigma(j)}^*(x)x_{\sigma(j)}\|\leq\|x-\sum_{j\geq 1}^{N} x_j^*(x)x_j\|+\|\sum_{j\leq m,\ \sigma(j)\geq N} x_{\sigma(j)}^*(x)x_{\sigma(j)}\|$$

$$=\|x-\sum_{j=1}^{N} x_j^*(x)x_j\|+\|\sum_{k\geq N+1}^{M_0}\beta_k x_k^*(x)x_k\|=\mathrm{I}+\mathrm{II} \tag{7.12}$$

where $M_0=\max\{\sigma(1),\dots,\sigma(m)\}$ and

$$\beta_j=\left\{\begin{array}{ll}1 & \text{if } k=\sigma(j) \text{ for some } j\leq m\\ 0 & \text{otherwise}\end{array}\right\}$$

By our hyphotesis,

$$\mathrm{II} \leq \left\| \sum_{k=N+1}^{M_0} \beta_k x_k^*(x) x_k \right\| = \left\| S_\beta \left(\sum_{k=N+1}^{M_0} x_k^*(x) x_k \right) \right\| \leq C \left\| \sum_{k=N+1}^{M_0} x_k^*(x) x_k \right\|$$

$$\leq C \left\{ \left\| x - \sum_{k=1}^{M_0} x_k^*(x) x_k \right\| + \left\| x - \sum_{k=1}^{N} x_k^*(x) x_k \right\| \right\} \overset{(7.12)}{\leq}$$

$$C \left[\frac{\varepsilon}{2C+1} + \frac{\varepsilon}{2C+1} \right] = \frac{2C\varepsilon}{2C+1}$$

Replacing in (7.12) and using (7.11) we conclude

$$\left\| x - \sum_{j=1}^{m} x_{\sigma(j)}^*(x) x_{\sigma(j)} \right\| \leq \frac{\varepsilon}{2C+1} + \frac{2C\varepsilon}{2C+1} = \varepsilon \quad \text{for all } m \geq M$$

QED

The above results give the following working characterizations for unconditional bases.

Theorem 7.8. Let $\mathcal{B} = \{x_n\}_{n=1}^\infty$ be a basis of a Banach space $(\mathbb{B}, \|\ \|)$. The following statements are equivalent:

(i) $\mathcal{B}$ is an unconditional basis for $\mathbb{B}$

(ii) There exists a constant $C < \infty$ such that $\forall x \in X$, $\|S_\beta(x)\| \leq C\|x\|$ for all sequences $\beta = \{\beta_n\}_{n=1}^\infty \subset \mathbb{F}$ with $|\beta_n| \leq 1$.

(iii) There exists a constant $C < \infty$ such that $\forall x \in X$, $\|S_\varepsilon(x)\| \leq C\|x\|$ for all sequences $\varepsilon = \{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n = \pm 1$.

(iv) There exists a constant $C < \infty$ such that $\forall x \in X$, $\|S_\beta(x)\| \leq C\|x\|$ for all finitely non-zero sequences $\beta = \{\beta_n\}_{n=1}^\infty$ with $\beta_n = 1$ or 0

Proof. (i) $\Rightarrow$ (ii) is Lemma 7.7

(ii) $\Rightarrow$ (iii) is obvious

(iii) $\Rightarrow$ (iv) Choose $\beta = \{\beta_n\}_{n=1}^\infty$ with $\beta_n = 1$ or 0.

Define
$$\varepsilon_n = \begin{cases} 1 & \text{if } \beta_n = 1 \\ -1 & \text{if } \beta_n = 0 \end{cases}$$

Then
$$\sum_{n=1}^{\infty} \varepsilon_n x_n^*(x) x_n = 2\sum_{n=1}^{\infty} \beta_n x_n^*(x) x_n - \sum_{n=1}^{\infty} x_n^*(x) x_n$$

Hence,
$$\|S_\beta(x)\| = \frac{1}{2}\| S_\varepsilon(x) + S_{\mathbb{1}}(x)\| \leq \frac{1}{2} 2C\|x\| = C\|x\|.$$

(iv) $\Rightarrow$ (i) is lemma 7.7 QED

Exercise 7.1 Let $\beta = \{x_n\}_{n=1}^{\infty}$ be a basis in a Banach space $(\mathbb{B}, \|\ \|)$. Show that the following two conditions are equivalent.

1. β is an unconditional basis of $\mathbb{B}$
2. There exists a constant $1 \leq k < \infty$ such that for all $N \in \mathbb{N}$
$$\Big\| \sum_{n=1}^{N} \varepsilon_n a_n x_n \Big\| \leq k \Big\| \sum_{n=1}^{N} a_n x_n \Big\| \qquad (7.13)$$

For all $a_1, \dots a_N \in \mathbb{F}$ and any choice of signs $\varepsilon_1, \dots, \varepsilon_N$.

See pages 7.21 an 7.22 for more equivalent conditions and a more detailed proof.

S/ 1 $\Rightarrow$ 2. We use (iv) of Theorem 7.8. Choose $\varepsilon := \{\varepsilon_n\}_{n=1}^{\infty}$ given by
$$\varepsilon_n = \begin{cases} \varepsilon_n & \text{if } 1 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

By (iii) of Theorem 7.8, if $x = \sum_{n=1}^{N} a_n x_n$, then
$$S_\varepsilon(x) = \sum_{n=1}^{N} \varepsilon_n a_n x_n$$

and
$$\Big\| \sum_{n=1}^{N} \varepsilon_n a_n x_n \Big\| = \| S_\varepsilon(x)\| \leq C\|x\| = C \Big\| \sum_{n=1}^{N} a_n x_n \Big\|.$$

2. ⇒ 1. Take a convergent series $\sum_{n=1}^{\infty} a_n x_n$ in $(\mathbb{B}, \|\ \|)$.
We are going to show that for any increasing subsequence of non negative integers $n_1 < n_2 < \ldots < n_k < \ldots$ the series $\sum_{k=1}^{\infty} a_{n_k} x_{n_k}$ converges in $(\mathbb{B}, \|\ \|)$. This will show the result by the equivalence $(i) \Leftrightarrow (iii)$ in Theorem 7.5
Enough to prove that $\left\{ \sum_{k=1}^{m} a_{n_k} x_{n_k} \right\}_{m=1}^{\infty}$ is a Cauchy sequence, since $(\mathbb{B}, \|\ \|)$ is complete.

Given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that if $m_2 > m_1 \geq N$,

$$\left\| \sum_{n=m_1+1}^{m_2} a_n x_n \right\| \leq \frac{\varepsilon}{K}$$

By hypothesis, if $N \leq n_{k+1} < \ldots < n_{k+\ell}$ we have,

$$\left\| \sum_{j=k+1}^{k+\ell} a_{n_j} x_{n_j} \right\| \leq 2K \left\| \sum_{j=n_{k+1}}^{n_{k+\ell}} a_j x_j \right\| \leq \varepsilon$$

which proves the desired result.

Def 7.9 Let $\beta = \{x_n\}_{n=1}^{\infty}$ be an unconditional basis of a Banach space $(\mathbb{B}, \|\ \|)$. The unconditional basis constant $K_u(\beta)$ is the least constant K for which (7.13) holds.

Let $\beta = \{x_n\}_{n=1}^{\infty}$ be an unconditional basis of a Banach space $(\mathbb{B}, \|\ \|)$. For $A \subset \mathbb{N}$ define

$$P_A : X \longrightarrow X,$$

by

$$P_A(x) = P_A\left(\sum_{n=1}^{\infty} x_n^*(x) x_n\right) = \sum_{n \in A} x_n^*(x) x_n \qquad (7.14)$$

The operator P_A is well defined by Theorem 7.8, it is a linear projection, and $\|P_A(x)\| \leq C \|x\|$.

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Define $\varepsilon_n = \begin{Bmatrix} 1 & \text{if } \beta_n = 1 \\ -1 & \text{if } \beta_n = 0 \end{Bmatrix}$.

Then
$$\sum_{n=1}^{\infty} \varepsilon_n x_n^*(x) x_n = 2 \sum_{n=1}^{\infty} \beta_n x_n^*(x) x_n - \sum_{n=1}^{\infty} x_n^*(x) x .$$

Hence,
$$\| S_\beta(x) \| = \frac{1}{2} \| S_\varepsilon(x) + S_1(x) \| \leq \frac{1}{2} 2C \|x\| = C\|x\|$$

(iv) $\Rightarrow$ (i) is lemma 7.7 QED

Exercise 7.1 Let $\mathcal{B} = \{x_n\}_{n=1}^{\infty}$ be a basis in a Banach space $(\mathbb{B}, \| \|)$. Show that the following three conditions are equivalent

1. $\mathcal{B}$ is an unconditional basis of $(\mathbb{B}, \| \|)$
2. There exists K, $1 \leq K < \infty$ such that for all $N \in \mathbb{N}$,
$$\left\| \sum_{n=1}^{N} \varepsilon_n a_n x_n \right\| \leq K \left\| \sum_{n=1}^{N} a_n x_n \right\| \qquad (7.3)$$
for all $a_1, \dots, a_n \in \mathbb{F}$ and any choice of signs $\varepsilon_n = \pm 1$.
3. There exists K, $1 \leq K < \infty$ such that for all $N \in \mathbb{N}$,
$$\left\| \sum_{n=1}^{\infty} \beta_n a_n x_n \right\| \leq K \left\| \sum_{n=1}^{N} a_n x_n \right\|$$
for all $a_1, \dots a_n \in \mathbb{F}$ and all $\beta = (\beta_n)_{n=1}^{N}$ with $\beta_n = 0$ or 1.

---

1. $\Rightarrow$ 2. We use (iii) of Theorem 7.8. If $x = \sum_{n=1}^{N} a_n x_n$,
$$S_\varepsilon(x) = \sum_{n=1}^{N} \varepsilon_n a_n x_n$$
and
$$\left\| \sum_{n=1}^{N} \varepsilon_n a_n x_n \right\| = \| S_\varepsilon(x) \| \leq C \|x\| = C \left\| \sum_{n=1}^{N} a_n x_n \right\|$$

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$2 \Rightarrow 3$ Given $\beta_n = 0$ or 1, if $n = 1, 2, \ldots N$, define

$$\varepsilon_n = \begin{cases} 1 & \text{if } \beta_n = 1 \\ -1 & \text{if } \beta_n = 0 \end{cases}, \quad n = 1, 2, \ldots, N.$$

Then

$$\sum_{n=1}^{N} \varepsilon_n a_n x_n = 2 \sum_{n=1}^{N} \beta_n a_n x_n - \sum_{n=1}^{N} a_n x_n$$

Hence

$$\left\| \sum_{n=1}^{N} \beta_n a_n x_n \right\| = \frac{1}{2} \left\| \sum_{n=1}^{N} \varepsilon_n a_n x_n + \sum_{n=1}^{N} a_n x_n \right\| \le$$

$$\le \frac{1}{2} \left(\left\| \sum_{n=1}^{N} \varepsilon_n a_n x_n \right\| + \left\| \sum_{n=1}^{N} a_n x_n \right\| \right) = k.$$

$3 \Rightarrow 1$. Take a convergent series $\sum_{n=1}^{\infty} a_n x_n$ in $(\mathbb{B}, \|\ \|)$. We are going to show that for any increasing sequence of non-negative integers $n_1 < n_2 < \ldots < n_k < \ldots$, the series $\sum_{k=1}^{\infty} a_{n_k} x_{n_k}$ converges in $(\mathbb{B}, \|\ \|)$. This will show the result by the equivalence of (i) and (ii) in Theorem 7.5. It is enough to prove that

$$\left\{ \sum_{k=1}^{m} a_{n_k} x_{n_k} \right\}_{m=1}^{\infty}$$

is a Cauchy sequence in $(\mathbb{B}, \|\ \|)$, since $(\mathbb{B}, \|\ \|)$ is complete.

Since $\sum_{n=1}^{\infty} a_n x_n$ converges, given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that if $m_2 > m_1 \ge N$,

$$\left\| \sum_{n=m_1+1}^{m_2} a_n x_n \right\| \le \frac{\varepsilon}{k}.$$

Using item 3, if $n_{k+\ell} > \ldots > n_{k+1} \ge N$ we have

$$\left\| \sum_{j=k+1}^{k+\ell} a_{n_j} x_{n_j} \right\| = \left\| \sum_{n=n_{k+1}}^{n_{k+\ell}} \beta_n a_n x_n \right\| \le k \left\| \sum_{n=n_{k+1}}^{n_{k+\ell}} a_n x_n \right\| \le k \cdot \frac{\varepsilon}{k} = \varepsilon,$$

where

$$\beta_n = \begin{cases} 1 & \text{if } n = n_{k+s},\ 1 \le s \le \ell \\ 0 & \text{otherwise} \end{cases}.$$

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Def 7.9 Let $\beta = \{x_n\}_{n=1}^{\infty}$ be an unconditional basis of a Banach space $(\mathbb{B}, \|\ \|)$. The unconditional basis constant, $k_u(\beta)$, is the least constant k for wich (7.3) holds for all $a_1, \dots, a_n \in \mathbb{F}$ and all $\varepsilon_n = \pm 1$, $n = 1, \dots, N$.

Example 7.10 For $1 \le p < \infty$, the set $\mathcal{E} = \{\delta_n\}_{n=1}^{\infty}$ is a basis for $\ell^p = \ell^p(\mathbb{N})$ (see Example 6.10). It is also unconditional. To see this, we prove item (iii) of theorem 7.8. If $\varepsilon = \{\varepsilon_n\}_{n=1}^{\infty}$ with $\varepsilon_n = \pm 1$ and

$z = \sum_{n=1}^{\infty} z_n \delta_n \in \ell^p$

$$\|S_\varepsilon(z)\|_{\ell^p} = \|S_\varepsilon(\sum_{n=1}^{\infty} z_n \delta_n)\|_{\ell^p} = \|\sum_{n=1}^{\infty} \varepsilon_n z_n \delta_n\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |\varepsilon_n z_n|^p\right)^{1/p}$$

$$= \left(\sum_{n=1}^{\infty} |z_n|^p\right)^{1/p} = \|x\|_{\ell^p}$$

Therefore, $\mathcal{E}$ is an unconditional basis of ℓ^p, $1 \le p < \infty$, and $k_u(\mathcal{E}) = 1$.

Exercise 7.B Show that $\mathcal{E} = \{\delta_n\}_{n=1}^{\infty}$ is an unconditional basis of c_0 (see Example 6.11) and find $k_u(\mathcal{E})$ in c_0.

S/ Similar to Example 7.10. Also $k_u(\mathcal{E}) = 1$

Exercise 7.C Show that an orthonormal basis $\beta = \{e_n\}_{n=1}^{\infty}$ in a Hilbert space $(\mathbb{H}, \langle\ ,\ \rangle)$ is unconditional and find $k_u(\beta)$ (see Example 6.9).

S/ We prove item (iii) of Theorem 7.8. If $\varepsilon = \{\varepsilon_n\}_{n=1}^{\infty}$ with $\varepsilon_n = \pm 1$, and $x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$, by Plancherel identity (Thm 5.11)

$$\|S_\varepsilon(x)\|^2 = \|\sum_{n=1}^{\infty} \varepsilon_n \langle x, e_n\rangle e_n\|^2 = \sum_{n=1}^{\infty} |\varepsilon_n \langle x, e_n\rangle|^2 =$$

$$= \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = \|x\|^2.$$

A similar argument shows that $\|P_A(x)\| \le \|x\|$ for all $A \subset \mathbb{N}$. Since $\|P_{\mathbb{N}}(x)\| = \|x\|$ we obtain $k_{su}(\beta) = 1$. Also $k_u(\beta) = 1$.

Example 7.11 Consider the sum basis $\mathcal{F} = \{f_n\}_{n=1}^{\infty}$, with $f_n = \delta_1 + \dots + \delta_n$, of c_0 (see Example 6.D). We will show that $\mathcal{F}$ is conditional in c_0, that is, $\mathcal{F}$ is not unconditional

Fix $N = 2n$ even. Let $z_N = (f_{2n} - f_{2n-1}) + (f_{2n-2} - f_{2n-3}) + \dots + f_2 - f_1 = \delta_{2n} + \delta_{2n-2} + \dots + \delta_2$. We have $\|z_N\|_\infty = 1$. Let $\beta = \{\beta_n\}_{n=1}^{\infty}$ such that $\beta_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$. Then

$$S_\beta(z_N) = f_{2n} + f_{2n-2} + \dots + f_2 = \sum_{j=1}^{2n} \delta_j + \sum_{j=1}^{2n-2} \delta_j + \dots + \sum_{j=1}^{2} \delta_j$$

$$= \frac{N}{2}\delta_1 + \frac{N}{2}\delta_2 + \dots + \delta_{2n-1} + \delta_{2n}$$

therefore, $\|S_\beta(z_N)\|_\infty = \frac{N}{2}$. Hence $\frac{\|S_\beta(z_N)\|_\infty}{\|z_N\|_\infty} = \frac{N}{2}$ and

$$\|S_\beta\| \ge \sup_{z_N} \frac{\|S_\beta(z_N)\|_\infty}{\|z_N\|_\infty} = \infty$$

Hence, item (ii) of theorem 8.7 does not hold. Consequently, $\mathcal{F}$ is a conditional basis in c_0

Exercise 7.D Let c be the set of all sequences $z = \{z_n\}_{n=1}^{\infty} \in \ell^\infty$ such that $\lim_{n\to\infty} z_n$ exists.) Let $s_n = \sum_{j=n}^{\infty} \delta_n$

Show that $\mathcal{S} = \{s_n\}_{n=1}^{\infty}$ is a monotone and normalized basis in c, but it is not unconditional. This is called the summing basis in [LT] (page 20).

S/ Start proving that $\mathcal{S}=\{s_n\}_{n=1}^{\infty}$ is a basis for c. Let $z=(z_n)_{n=1}^{\infty}$ and $\lim_{n\to\infty} z_n = z_0$. We want to obtain an expansion of the form $z = \sum_{n=1}^{\infty} a_n s_n$. What is the relation between the a_n's and z_n's? From

$$z = \sum_{n=1}^{\infty} a_n s_n = \sum_{n=1}^{\infty} a_n \left(\sum_{j=n}^{\infty} \delta_j \right) = \sum_{j=1}^{\infty} \left(\sum_{n=1}^{j} a_n \right) \delta_j \qquad (7.15)$$

we deduce the system of equations

$$z_j = \sum_{n=1}^{j} a_n \quad , j=1,2,\dots$$

For $j=1$ this implies $a_1 = z_1$; for $j=2$, $z_2 = a_2 + a_1$ implies $a_2 = z_2 - z_1$. If we asume $a_k = z_k - z_{k-1}$, $k=2,\dots,n-1$, then

$$z_n = \sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n-1} a_k + a_n = z_1 + \sum_{k=2}^{n-1} (z_k - z_{k-1}) + a_n$$

$$= z_{n-1} + a_n \Rightarrow a_n = z_n - z_{n-1}.$$

This induces us to prove

$$z = z_1 s_1 + \sum_{n=2}^{\infty} (z_n - z_{n-1}) s_n \qquad (7.16)$$

with convergence in ℓ^{∞}. First observe that for $N \in \mathbb{N}$

$$z_1 s_1 + \sum_{n=2}^{N} (z_n - z_{n-1}) s_n = \sum_{n=1}^{N-1} z_n \delta_n + (z_N - z_{N-1}) \sum_{n=N}^{\infty} \delta_n$$

Then:

$$\left\| z - \left(z_1 s_1 + \sum_{n=2}^{N} (z_n - z_{n-1}) s_n \right) \right\|_{\infty} = \left\| \sum_{n=N}^{\infty} z_n \delta_n - (z_N - z_{N-1}) \sum_{n=N}^{\infty} \delta_n \right\|_{\infty}$$

$$= \left\| \sum_{n=N+1}^{\infty} (z_n - z_N) \delta_n \right\|_{\infty} = \sup_{n \geq N+1} |z_n - z_N|$$

and

$$\lim_{N\to\infty} \left\| z - \left(z_1 s_1 + \sum_{n=2}^{N} (z_n - z_{n-1}) s_n \right) \right\| = \lim_{N\to\infty} \sup_{n \geq N+1} |z_n - z_N| = 0$$

since $\lim_{n\to\infty} z_n = z_0$ exist. The expression (7.16) is unique.

For $z = \{z_n\}_{n=1}^{\infty} \in C$, from

$$0 = z_1 S_1 + \sum_{n=2}^{\infty} (z_n - z_{n-1}) S_n = \sum_{n=1}^{\infty} z_n \delta_n$$

we deduce $z_n = 0$ for all $n = 1, 2, 3, \ldots$, which implies $z_1 = 0$ and $z_n - z_{n-1} = 0$ if $n = 2, 3, \ldots$

It is normalized because $\| S_n \|_\infty = \| \sum_{j=n}^{\infty} \delta_j \|_\infty = 1$.

From (7.15), if $z = \sum_{n=1}^{N} a_n S_n$ we deduce

$$\|z\|_\infty = \sup_{1 \le j \le N} \left| \sum_{n=1}^{j} a_n \right| \qquad (7.17)$$

Then

$$\| S_N \|_{op} = \sup_{\substack{z \neq 0 \\ z \in C}} \frac{\| S_N(z) \|_\infty}{\|z\|_\infty} = \sup_{\substack{z \neq 0 \\ z \in C}} \frac{\| z_1 S_1 + \sum_{n=2}^{N} (z_n - z_{n-1}) S_n \|_\infty}{\|z\|_\infty}$$

Now, by (7.17)

$$\| z_1 S_1 + \sum_{n=2}^{N} (z_n - z_{n-1}) S_n \|_\infty = \sup \{ |z_1|, |z_2|, |z_3|, \ldots |z_n| \} \le \|z\|_\infty.$$

Therefore,

$$\| S_N \|_{op} \le 1$$

Hence $k_b(\mathcal{S}) \le 1$. On the other hand $\| S_1 \|_{op} \ge \frac{\| S_1(S_1) \|_\infty}{\| S_1 \|_\infty} = 1$.

Therefore, $\mathcal{S}$ is monotone.

It remains to show that it is not unconditional. Let

$$z^{(N)} = (S_{2N} - S_{2N-1}) + (S_{2N-2} - S_{2N-3}) + \ldots + (S_2 - S_1)$$

$$= \sum_{j=1}^{N} (S_{2j} - S_{2j-1}) = \sum_{j=1}^{N} \delta_{2j}$$

We have $\| z^{(N)} \|_\infty = 1$ for all $N = 1, 2, \ldots$

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Let $\varepsilon = (\varepsilon_n)_{n=1}^{\infty}$ be the sequence $\varepsilon_n = 1$ if if n is even and $\varepsilon_n = -1$ if n is odd. Then

$$S_\varepsilon(z^{(N)}) = \sum_{j=1}^{2N} s_j'$$

and from (7.17),

$$\|S_\varepsilon(z^{(N)})\|_\infty = \sup_{1\le j\le 2N} \left|\sum_{n=1}^{j} 1\right| = 2N$$

Therefore

$$\|S_\varepsilon\|_{op} \geq \frac{\|S_\varepsilon(z^{(N)})\|_\infty}{\|z^{(N)}\|_\infty} = 2N \longrightarrow \infty \text{ as } N \to \infty$$

By theorem 7.8, $\mathcal{S}$ is not an unconditional basis in c.