6. Bases and Schauder Bases

Let $V=(V,+,\cdot)$ be a vector space over $\mathbb{F}$ ($=\mathbb{R}$ or $\mathbb{C}$). A set $\mathcal{B}=\{u_1,\dots,u_n\}\subset V$ is a basis for V if $\{u_1,\dots,u_n\}$ are linearly independent (l.i.) (i.e. if for some $\alpha_1,\dots,\alpha_n\in\mathbb{F}$, $\sum_{j=1}^{n}\alpha_j u_j=0$, then $\alpha_1=\dots=\alpha_n=0$) and for every $x\in V$, there exist $\alpha_1,\dots,\alpha_n$ such that

$$x=\sum_{j=1}^{n}\alpha_j u_j \qquad (6.1)$$

Linear independence of $\mathcal{B}$ implies that the coefficients α_j, $j=1,\dots,n$, in (6.1) are unique. It is a basic result in elementary linear algebra that if V has a basis with n elements, any other basis will have the same number of elements. This number n will be called the dimension of V. In this case, V is said to be a finite dimensional vector space.

Lemma 6.1. Let $(V,+,\cdot)$ be a vector space over $\mathbb{F}$. Let $\mathcal{B}=\{u_1,\dots,u_n\}\subset V$. The following are equivalent:

1. $\mathcal{B}$ is a basis for V
2. For every $x\in V$ there exists $\alpha_1,\dots,\alpha_n\in\mathbb{F}$, unique, such that $$x=\sum_{j=1}^{n}\alpha_j u_j \qquad (6.2)$$

P/ $1.\Rightarrow 2$, has been commented above.
$2\Rightarrow 1$. Since $\sum_{j=1}^{n}0u_j=0=\sum_{j=1}^{n}\alpha_j u_j$, if the α_j are unique, we deduce $\alpha_j=0$, $j=1,\dots,n$, which shows that $\mathcal{B}$ is l.i.

Condition 2 in Lemma 6.1 seems suitable to extend the definition of basis to infinite dimensional vector spaces, that is, those that do not have finite dimensional bases. But (6.2) will be an infinite sum and we need to have a topology in V to be able to talk about convergence.

Def 6.2. A topological vector space $(V, \mathcal{T})$ is a vector space $(V, +, \cdot) = V$ with a topology $\mathcal{T}$ such that the operations $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$ are continuous.

Every normed vector space is topological, where the topology is generated by the open balls $\{B_r(x) : x \in V, r > 0\}$. In particular, Banach spaces and Hilbert spaces are topological vector spaces.

In the case of Hilbert spaces $(\mathbb{H}, \langle , \rangle)$ we have defined in Def 5.12 the concept of orthonormal bases. By Theorem 5.11, $\beta = \{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $(\mathbb{H}, \langle , \rangle)$ if and only if

$$\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n = x \qquad \text{for all } x \in \mathbb{H} \tag{6.3}$$

where the convergence of the partial sums is in the norm $\|x\| = \sqrt{\langle x, x \rangle}$. It was observe in Exercise 5.4 that the coefficients are unique.

This is an idea that can be used to define a basis in a topological vector space

Def 6.3 Let $(V, \mathcal{T})$ be a topological vector space. A countable ordered set $\beta = \{x_n\}_{n=1}^{\infty} \subset V$ is a basis for V if for every $x \in V$ there exist $\{\alpha_n(x)\}_{n=1}^{\infty} \subseteq \mathbb{F}$, unique, such that

$$x = \sum_{n=1}^{\infty} \alpha_n(x) x_n \tag{6.4}$$

where the convergence is in the topology $\mathcal{T}$, that is

$$\lim_{N \to \infty} \sum_{n=1}^{N} \alpha_n(x) x_n \overset{\mathcal{T}}{=} x$$

Warning. In Definition 6.3 we have chosen $\mathbb{N}$ to ordered the elements of β. But any other countable set can be used, for example $\mathbb{Z}$. In this case, the order of the elements has to be specified; for example

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$$

The order of the basis is important in Def 6.3. If the elements of the basis are permuted, the new set may not be a basis for V, in general. We will allways write $\sum_{n=1}^{\infty}$ because we are interested in infinite dimensional Banach spaces, but our results will hold for N-dimensional spaces, replacing $\sum_{n=1}^{\infty}$ by $\sum_{n=1}^{N}$, in which case the convergence in (6.4) is an equality as in (6.1)

Note for the reader. There is a notion of bases in a vector space V, which is usually called Hamel basis. A Hamel basis in a vector space V is a collection $\beta \subset V$ such that

i) Every finite subset of β is linearly independent

ii) For every $x \in V$, there exists a finite set $F \subset \beta$ such that
$x = \sum_{u \in \beta} \alpha_u u$ for some $\alpha_u \in \mathbb{F}$.

From the Baire category theorem it can be deduced that a Hamel basis in an infinite dimensional Banach space has to be uncountable. We will not consider Hamel bases in this note.

Remark. When consider bases in a Banach space $(\mathbb{B}, \| \ \|)$, equality (6.4) means convergence in norm of the partial sums, that is

$$\lim_{N \to \infty} \left\| x - \sum_{n=1}^{N} \alpha_n x_n \right\| = 0 \qquad (6.5)$$

If a Banach space $(\mathbb{B}, \| \ \|)$ has a basis $\beta = \{x_n\}_{n=1}^{\infty}$, then $\mathbb{B}$ is separable, since the set of all finite linear combinations $\sum_{n=1}^{N} r_n x_n$, with r_n rational (real or complex) numbers and $N=1,2,3,\ldots$ is a countable dense set in $\mathbb{B}$.

We have noted just before Theorem 5.13 that every separable Hilbert space has a basis (in fact, an orthonormal basis). The question if every separable Banach space has a basis is known as the basis problem, and it was solved in the negative by P. Enflo (A counterexample to the approximation property of Banach spaces, Acta Math. 130, 309-317 (1973).).

Let $\beta=\{x_n\}_{n=1}^{\infty}$ be a basis in a topological space. The set β is linearly independent because if $F\subset\beta$ is finite, $\sum_{x_n\in F}\alpha_n\cdot x_n=0$ implies $\alpha_n=0$ due to uniqueness in (6.4). In particular $x_n\neq 0$ for all $x_n\in\beta$.

By (6.4) to each $x\in V$ we can define

$$f_n : V \to \mathbb{F} \qquad n=1,2,3,\ldots \qquad (6.6)$$
$$x \to f_n(x) := \alpha_n(x)$$

These maps are clearly linear (again because of uniqueness in (6.4).

Def 6.4 A basis $\beta=\{x_n\}_{n=1}^{\infty}$ of a topological vector space $(V,\mathcal{T})$ is called a Schauder basis if the maps given in (6.6) are all continuous.

The name Schauder basis is given in honor of J. Schauder who introduced the concept of basis in infinite dimensional spaces in 1927.

Our first important result will be to show that any basis in a (separable) Banach space is a Schauder basis. That is, if $(\mathbb{B},\|\ \|)$ is the Banach space, we will show that $f_n\in\mathbb{B}^*$ (the dual of $\mathbb{B}$) for all $n\in\mathbb{N}$.

To do this, we define on $\mathbb{B}$ a new norm and prove that is equivalent to the original one

Lemma 6.5 Let $(\mathbb{B}, \|\ \|)$ be a Banach space over a field $\mathbb{F}$ $(=\mathbb{R}$ or $\mathbb{C})$ and $\beta = \{x_n\}_{n=1}^{\infty}$ be a basis for $\mathbb{B}$. If $x = \sum_{n=1}^{\infty} \alpha_n(x) x_n$ is the unique representation of x in terms of the ordered elements of β, then

$$|||x||| := \sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^{N} \alpha_n(x) x_n \right\|$$

defines a norm in $\mathbb{B}$ that is equivalent to $\|\ \|$.

Recall Two norm, $\|\ \|_1$ and $\|\ \|_2$, for the same space vector space V are equivalent if there exists $C_1, C_2 < \infty$ such that for all $x \in V$

$$C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1.$$

P/ Since the representation $x = \sum_{n=1}^{\infty} \alpha_n(x) x_n$ for the basis β is convergent in norm, the partial sums $S_N(x) = \sum_{n=1}^{N} \alpha_n(x) x_n$ are bounded. Hence $|||x||| < \infty$. Moreover, if $x \neq 0$

$$|||x||| = \sup_{N \in \mathbb{N}} \|S_N(x)\| \geq \lim_{N\to\infty} \|S_N(x)\| = \|x\| > 0 \qquad (6.7)$$

The other properties of the norm are easy to verify for $|||\ |||$.

Exercise 6.A In the last equality of (6.7) we have used that if $\lim_{n\to\infty} \|x - y_n\| = 0$, then $\lim_{n\to\infty} \|y_n\| = \|x\|$. Prove this result.

S/ $|\|x\| - \|y_n\|| \leq \|x - y_n\|$ by the reverse triangle inequality. Therefore, given $\varepsilon > 0$, if $\|x - y_n\| \leq \varepsilon$ for all $n \geq N(\varepsilon)$, then $|\|x\| - \|y_n\|| \leq \varepsilon$ also for all $n \geq N(\varepsilon)$.

(Replacing x_n by $x_n/\|x_n\|$, $\beta' = \left\{ \frac{x_n}{\|x_n\|} \right\}_{n=1}^{\infty}$ is also a basis for $(\mathbb{B}, \|\ \|)$ and $|||x|||$ does not change. (See exercise 6.B below))

We claim that $(\mathbb{B}, ||| \;|||)$ is complete. Choose $\{y^{(k)}\}_{k=1}^{\infty}$ a Cauchy sequence in $\mathbb{B}$ with respect to $||| \;|||$. Our goal is to find $y \in \mathbb{B}$ such that $y = \lim_{k\to\infty} y^{(k)}$ in $||| \;|||$.

For $\varepsilon > 0$, choose $N(\varepsilon) \in \mathbb{N}$ such that

$$|||y^{(k)} - y^{(m)}||| = \sup_{N \in \mathbb{N}} \left\| \sum_{j=1}^{N} (\alpha_j(y^{(k)}) - \alpha_j(y^{(m)}))x_j \right\| \le \varepsilon. \tag{6.8}$$

for all $k, m \ge N(\varepsilon)$. Then, for $n \in \mathbb{N}$,

$$|\alpha_n(y^{(k)}) - \alpha_n(y^{(m)})| = \frac{1}{\|x_n\|} \|(\alpha_n(y^{(k)}) - \alpha_n(y^{(m)}))x_n\|$$

$$= \frac{1}{\|x_n\|} \left\| \sum_{j=1}^{n} (\alpha_j(y^{(k)}) - \alpha_j(y^{(m)}))x_j - \sum_{j=1}^{n-1} (\alpha_j(y^{(k)}) - \alpha_j(y^{(m)}))x_j \right\|$$

$$\le \frac{1}{\|x_n\|} \left\| \sum_{j=1}^{n} (\alpha_j(y^{(k)}) - \alpha_j(y^{(m)}))x_j \right\| + \left\| \sum_{j=1}^{n-1} (\alpha_j(y^{(k)}) - \alpha_j(y^{(m)}))x_j \right\|$$

$$\le \frac{1}{\|x_n\|} [\varepsilon + \varepsilon] = \frac{2\varepsilon}{\|x_n\|}$$

for all $k, m \ge N(\varepsilon)$. This shows that the numerical sequence $\{\alpha_n(y^{(k)})\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$. Since $(\mathbb{F}, |\;|)$ is complete, there exist $\alpha_n \in \mathbb{F}$ for each $n \in \mathbb{N}$, such that

$$\alpha_n = \lim_{k\to\infty} \alpha_n(y^{(k)}) \quad \text{for all } n \in \mathbb{N}.$$

From (6.8) we obtain, when $k, m \ge N(\varepsilon)$

$$\left\| \sum_{j=1}^{n} (\alpha_j(y^{(k)}) - \alpha_j(y^{(m)}))x_j \right\| \le \varepsilon$$

Letting $m \to \infty$ we deduce for $k \ge N(\varepsilon)$ and $n \in \mathbb{N}$

$$\left\| \sum_{j=1}^{n} (\alpha_j(y^{(k)}) - \alpha_j)x_j \right\| \le \varepsilon. \tag{6.9}$$

Since for $n, l \in \mathbb{N}$, and $k \in \mathbb{N}$,

$$\sum_{j=n+1}^{n+\ell} \alpha_j x_j = \sum_{j=1}^{n+\ell} \alpha_j x_j - \sum_{j=1}^{n} \alpha_j x_j$$

$$= \sum_{j=1}^{n+\ell} (\alpha_j - \alpha_j(y^{(k)})) x_j - \sum_{j=1}^{n} (\alpha_j - \alpha_j(y^{(k)})) x_j + \sum_{j=n+1}^{n+\ell} \alpha_j(y^{(k)}) x_j$$

from (6.8) we obtain, for $k > N(\varepsilon)$ and $n, \ell \in \mathbb{N}$

$$\left\| \sum_{j=n+1}^{n+\ell} \alpha_j x_j \right\| \leq \varepsilon + \varepsilon + \left\| \sum_{j=n+1}^{n+\ell} \alpha_j(y^{(k)}) x_j \right\| .$$

The series $\sum_{j=1}^{\infty} \alpha_j(y^{(k)}) x_j$ converges to $y^{(k)}$ in the norm $\| \ \|$ since β is a basis for $(\mathbb{B}, \| \ \|)$. Taking ℓ large enough

$$\left\| \sum_{j=n+1}^{n+\ell} \alpha_j x_j \right\| \leq 3\varepsilon .$$

Therefore, $\sum_{j=1}^{\infty} \alpha_j x_j$ is convergent in $\| \ \|$. Since $(\mathbb{B}, \| \ \|)$ is a Banach space, there exist $y \in \mathbb{B}$ such that

$$y = \sum_{j=1}^{\infty} \alpha_j x_j \quad (\text{convergence in } \| \ \|).$$

By (6.9), for $k \geq N(\varepsilon)$,

$$|||y - y^{(k)}||| = \sup_{N \in \mathbb{N}} \left\| \sum_{j=1}^{N} (\alpha_j(y^{k}) - \alpha_j) x_j \right\| \leq \varepsilon$$

This shows that $(\mathbb{B}, ||| \ |||)$ is complete, as claimed.

By (6.7), $\|x\| \leq |||x|||$. By the open mapping theorem, there exists $C > 0$ such that $|||x||| \leq C'\|x\|$. Thus, $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms in $\mathbb{B}$. Notice that to apply the open mapping theorem we need that $(\mathbb{B}, \| \ \|)$ and $(\mathbb{B}, ||| \ |||)$ are Banach spaces. QED

Exercise 6.B Show that if $\beta = \{x_n\}_{n=1}^{\infty}$ is a basis for $(\mathbb{B}, \|\ \|)$ then $\beta' = \left\{\frac{x_n}{\|x_n\|}\right\}_{n=1}^{\infty}$ is also a basis for $(\mathbb{B}, \|\ \|)$ and the norm $|||x|||$ defined in lemma 6.5 does not change if we replace β by β'

S/ Let $x \in \mathbb{B}$. Since :

$$x = \sum_{n=1}^{\infty} \alpha_n(x) x_n = \sum_{n=1}^{\infty} \alpha_n(x) \|x_n\| \frac{x_n}{\|x_n\|}$$

and $\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} \alpha_n(x) x_n \right\| = \lim_{N\to\infty} \left\| x - \sum_{n=1}^{\infty} \alpha_n(x) \|x_n\| \frac{x_n}{\|x_n\|} \right\|$

and $\{\alpha_n(x)\}$ are unique, so are $\{\alpha_n(x)\|x_n\|\}$. Thus, β' is a basis for $\mathbb{B}$. Now

$$|||x||| = \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \alpha_n(x) x_n \right\| = \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \alpha_n(x) \|x_n\| \frac{x_n}{\|x_n\|} \right\|$$

which shows that $|||x|||$ does not change if we use β or β'.

Please, revise the Open Mapping Theorem in W. Rudin (Functional Analysis, McGraw-Hill, (1973). We have used Corollary 2.12, part (c), of this book in our proof.

Theorem 6.6. If $\beta = \{x_n\}_{n=1}^{\infty}$ is a basis in a Banach space $(\mathbb{B}, \|\ \|)$, then β is also a Schauder basis for $(\mathbb{B}, \|\ \|)$.

P/ We need to prove that the coefficient functionals $f_n : \mathbb{B} \to \mathbb{F}$ are continuous for every $n \in \mathbb{N}$. Write $x = \sum_{j=1}^{\infty} \alpha_j(x) x_j$. Then, for $n \in \mathbb{N}$,

$$|f_n(x)| = |\alpha_n(x)| = |\alpha_n(x)| \left\| \frac{x_n}{\|x_n\|} \right\| = \frac{1}{\|x_n\|} \|\alpha_n(x) x_n\|$$

$$= \frac{1}{\|x_n\|} \left\| \sum_{j=1}^{n} \alpha_j(x) x_j - \sum_{j=1}^{n-1} \alpha_j(x) x_j \right\| \leq$$

$$\leq \frac{1}{\|x_n\|} \left[\left\| \sum_{j=1}^{n} \alpha_j(x) x_j \right\| + \left\| \sum_{j=1}^{n-1} \alpha_j(x) x_j \right\| \right]$$

$$\leq \frac{2}{\|x_n\|} \sup_{N \in \mathbb{N}} \left\| \sum_{j=1}^{N} \alpha_j(x) x_j \right\| = \frac{2}{\|x_n\|} |||x||| \qquad (6.10)$$

By lemma 6.5, $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms in $\mathbb{B}$. Hence, there exist $C>0$ such that $|||x||| \leq C' \|x\|$. By (6.10), $|f_n(x)| \leq \frac{2C}{\|x_n\|} \|x\|$, which shows that f_n (which is linear) is continuous by Theorem 4.2. QED

If $\beta = \{x_n\}_{n=1}^{\infty}$ is a basis for a Banach space $(\mathbb{B}, \|\ \|)$, the coefficient functionals $f_n(x) = \alpha_n(x)$ defined in (6.6) belong to $\mathbb{B}^*$ (the dual of $\mathbb{B}$) by Theorem 6.6. It is customary to use the notation $f_n = x_n^*$. Therefore, (6.4) becomes

$$x = \sum_{n=1}^{\infty} x_n^*(x)\, x_n \quad , \quad x \in \mathbb{B}, \qquad (6.11)$$

with convergence in $\|\ \|$. Moreover, the uniqueness of the representation (6.11) gives

$$x_n^*(x_k) = \begin{Bmatrix} 1 \text{ if } k=n \\ 0 \text{ if } k \neq n \end{Bmatrix} = \delta_{n,k} \qquad (6.12)$$

Let $\beta = \{x_n\}_{n=1}^{\infty}$ be a basis for a Banach space $(\mathbb{B}, \|\ \|)$. Define the partial sums $S_N : \mathbb{B} \to \mathbb{B}$ by

$$S_N(x) = S_N\left(\sum_{n=1}^{\infty} x_n^*(x) x_n\right) = \sum_{n=1}^{N} x_n^*(x)\, x_n \ , \quad N \in \mathbb{N}$$

By theorem 6.6, each x_n^* is bounded, that is $\forall x \in \mathbb{B}$

$|x_n^*(x)| \le C_n \|x\|$ for some constant C_n. Then S_N is bounded since, for all $x \in \mathbb{B}$

$$\|S_N(x)\| = \left\| \sum_{n=1}^{N} x_n^*(x)\, x_n \right\| \le \sum_{n=1}^{N} |x_n^*(x)|\, \|x_n\|$$
$$\le \left(\sum_{n=1}^{N} C_n \|x_n\| \right) \|x\|$$

and $M := \sum_{n=1}^{N} C_n \|x_n\| < \infty$. But the operator S_N are uniformly bounded; the $|||\ |||$ of Lemma 6.5 can be written as $|||x||| = \sup_{N \in \mathbb{N}} \|S_N(x)\|$. Thus, for all $N \in \mathbb{N}$,

$\|S_N(x)\| \le |||x|||$. Since $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms in $\mathbb{B}$, there exists C such that $|||x||| \le C\|x\|$. Hence $\|S_N(x)\| \le C\|x\|$ with C independent of N.

This allows us to give the following definition

Def 6.7 Let $\mathcal{B} = \{x_n\}$ be a basis for a Banach space $(\mathbb{B}, \|\ \|)$. Then, $\sup_{N \in \mathbb{N}} \|S_N\|$ is finite, and the number

$$K_b(\mathcal{B}) = K_b := \sup_{N \in \mathbb{N}} \|S_N\|$$

is called the basis constant.

Remark 1 Since $\|S_N(x_1)\| = \|x_1\|$ for all $N \in \mathbb{N}$, it follows that $\|S_N\| \ge 1$. In the optimal case, that $K_b(\mathcal{B}) = 1$, the basis $\mathcal{B}$ is called monotone.

Remark 2. A Banach space $(\mathbb{B}, \|\ \|)$ with a basis $\mathcal{B} = \{x_n\}_{n=1}^{\infty}$ can be renormed in such a way that the basis $\mathcal{B}$ is monotone with the new norm. Just consider

$$|||x||| = \sup_{N \in \mathbb{N}} \|S_N(x)\|, \quad x \in \mathbb{B}.$$

Then, $\|x\| \leq |||x||| \leq k_b(\mathcal{B})\|x\|$. Hence, $|||\cdot|||$ is equivalent to $\|\cdot\|$. Moreover

$$|||S_N(x)||| = \sup_{k\in\mathbb{N}} \|S_k(S_N(x))\| = \sup_{k=1,\dots,N} \left\|\sum_{j=1}^{k} x_n^*(x)x_n\right\|$$

$$\leq |||x||| \Rightarrow |||S_N||| \leq 1$$

On the other hand, $|||S_N(x_1)||| = |||x_1||| \Rightarrow |||S_N||| \geq 1$.
Hence, $|||S_N||| = 1$ and the basis constant of $\mathcal{B}$ with this norm is 1. Thus, $\mathcal{B}$ is monotone with the norm $|||\ |||$.

<u>Def 6.8</u> A sequence $\mathcal{B}:\{x_n\}_{n=1}^{\infty}$ in a Banach space $(\mathbb{B}, \|\ \|)$ is called a <u>basic sequence</u> if it is a basis for $E(\mathcal{B}) :\equiv \overline{\text{span}}\{x_n\}_{n=1}^{\infty}$.

<u>Exercise 6.C.</u> Assume that $\mathcal{B}=\{x_n\}_{n=1}^{\infty}$ is a basis in a Banach space $(\mathbb{B}, \|\ \|)$. Show that for every sequence of escalars $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{F}$ and all integer m, n such that $m \leq n$, it holds

$$\left\|\sum_{k=1}^{m} \alpha_k x_k\right\| \leq k_b \left\|\sum_{k=1}^{n} \alpha_k x_k\right\|$$

S/ If $S_N : \mathbb{B} \to \mathbb{B}$ denote the partial sums operator,

$$\left\|\sum_{k=1}^{m} \alpha_k x_k\right\| = \left\|S_m\left(\sum_{k=1}^{n} \alpha_k x_k\right)\right\| \leq \|S_m\| \left\|\sum_{k=1}^{n} \alpha_k x_k\right\|$$

$$\leq k_b \left\|\sum_{k=1}^{n} \alpha_k x_k\right\|.$$

Example 6.9 An orthonormal basis $\mathcal{B}=\{e_n\}_{n=1}^{\infty}$ on a Hilbert space $(\mathbb{H},\langle,\rangle)$ (See Def 5.12) is a basis in the sense of Def. 6.3 due to item (6) of Theorem 5.11 and Exercise 5.8.
Since $\mathcal{B}$ is orthonormal its partial sums $S_N:\mathbb{H}\to\mathbb{H}$ are bounded:

$$\|S_N(x)\|^2=\|S_N\big(\sum_{n=1}^{\infty}\langle x,e_n\rangle e_n\big)\|=\|\sum_{n=1}^{N}\langle x,e_n\rangle e_n\|^2 \underset{\mathcal{B}\text{ o.n}}{=}$$

$$=\sum_{n=1}^{N}|\langle x,e_n\rangle|^2\le\sum_{n=1}^{\infty}|\langle x,e_n\rangle|^2=\|x\|^2$$

where the last equality is due to Parseval's identity.
Hence $\|S_N\|_{op}\le 1$. Since $S_N(e_1)=e_1$, $\|S\|_{op}\ge\frac{\|S_N(e_1)\|}{\|e_1\|}=1$.
Hence

$$K_b(\mathcal{B})=\sup_{N\in\mathbb{N}}\|S_N\|_{op}=1$$

and $\mathcal{B}$ is a monotone basis of $(\mathbb{H},\langle,\rangle)$

Example 6.10 Let $1\le p<\infty$. The spaces $\ell^p=\ell^p(\mathbb{N})$ of all $z=(z_1,z_2,\dots,z_n,\dots)$, $z_j\in\mathbb{F}$, such that

$$\|z\|_p:=\Big(\sum_{n=1}^{\infty}|z_n|^p\Big)^{1/p}<\infty \qquad (6.13)$$

are Banach spaces (See Theorem 1.2.8).
For $n\in\mathbb{N}$ let δ_n be the sequence given by $\delta_n(k)=\begin{cases}1 & \text{if } k=n\\ 0 & \text{if } k\ne n\end{cases}$.
The elements δ_n belong to ℓ^p and $\|\delta_n\|_p=1$. Let

$$\mathcal{B}=\{\delta_n\}_{n=1}^{\infty}\subset\ell^p.$$

Every $z=(z_n)_{n=1}^{\infty}\in\ell^p$ can be written as $z=\sum_{n=1}^{\infty}z_n\delta_n$ and by (6.13)

$$\lim_{N\to\infty}\|z-\sum_{n=1}^{N}z_n\delta_n\|_p=\lim_{N\to\infty}\|\sum_{n=N+1}^{\infty}z_n\delta_n\|_p=$$

$$=\lim_{N\to\infty}\Big(\sum_{n=N+1}^{\infty}|z_n|^p\Big)^{1/p}=0.$$

The coefficient in the representation $z = \sum_{n=1}^{\infty} z_n \delta_n$ are obviously unique. Therefore, $\mathcal{E}$ is a basis for ℓ^p, for $1 \le p < \infty$, which is called the canonical basis of ℓ^p

Since

$$\|S_N(z)\|_p = \|S_N(\sum_{n=1}^{\infty} z_n \delta_n)\|_p = \|\sum_{n=1}^{N} z_n \delta_n\|_p$$

$$= \left(\sum_{n=1}^{N} |z_n|^p\right)^{1/p} \le \left(\sum_{n=1}^{\infty} |z|^p\right)^{1/p} = \|z\|_p$$

it follows that $\mathcal{E} = \{\delta_n\}_{n=1}^{\infty}$ is a monotone basis in ℓ^p.

Example 6.11 The set $\mathcal{E} = \{\delta_n\}_{n=1}^{\infty}$ defined in Example 6.10 is contained in $\ell^\infty = \ell^\infty(\mathbb{N}) = \{z = (z_n)_{n=1}^{\infty} : \|z\|_\infty := \sup_{n\in\mathbb{N}} |z_n| < \infty\}$, but it is not a basis of ℓ^∞. Although, formally, we can write $z = \sum_{n=1}^{\infty} z_n \delta_n$, this series does not always converge in ℓ^∞. To see this, let $\mathbb{1}$ be the sequence for which $z_n = 1$ for all $n \in \mathbb{N}$. Clearly, $\|\mathbb{1}\|_\infty = 1$. But

$$\lim_{N\to\infty} \|\mathbb{1} - \sum_{n=1}^{N} \delta_n\|_\infty = \lim_{N\to\infty} \|\sum_{n=N+1}^{\infty} \delta_n\|_\infty = 1$$

The set $\mathcal{E} = \{\delta_n\}_{n=1}^{\infty}$ is also contained in $c_0 = \{z = (z_n)_{n=1}^{\infty} : \|z\|_\infty < \infty \text{ and } \lim_{n\to\infty} |z_n| = 0\}$. Now, the collection $\mathcal{E} = \{\delta_n\}_{n=1}^{\infty}$ is a basis of c_0. The representation $z = \sum_{n=1}^{\infty} z_n \delta_n$ is unique and if $z = (z_n)_{n=1}^{\infty} \in c_0$

$$\lim_{N\to\infty} \|z - \sum_{n=1}^{N} z_n \delta_n\|_\infty = \lim_{N\to\infty} \|\sum_{n=N+1}^{\infty} \delta_n\| =$$

$$= \sup_{n \ge N+1} |z_n| \to 0$$

since $\lim_{n\to\infty} |z_n| = 0$.

Exercise 6.D Let c be the space of all sequences $z=(z_n)_{n=1}^{\infty}$ in ℓ^{∞} such that $\lim_{n\to\infty} z_n$ exists. Prove that $(c, \|\ \|_{\infty})$ is a Banach space. Let $e_0=(1,1,\dots,1,\dots)$ and $e_n=\delta_n$. Prove that $\{e_n\}_{n\geq 0}^{\infty}$ is a basis for c

S/ (Hint) Every $z\in c$ has a unique expansion

$$z=(\lim_{n\to\infty} z_n)\, e_0 + \sum_{j=1}^{\infty} (z_j - \lim_{n\to\infty} z_n)\, e_n$$

Exercise 6.E (Some bases in c_0) Let $\mathcal{B}=\{\delta_n\}_{n=1}^{\infty}$ be the canonical basis of $(c_0, \|\ \|_{\infty})$ (see Example 6.11). Define

$$f_n=\delta_1+\dots+\delta_n \quad , n\in\mathbb{N}$$

Show that $\mathcal{F}=\{f_n\}_{n=1}^{\infty}$ is a basis for $(c_0, \|\ \|_{\infty})$. Find its basis constant

S/ The idea is to show that for $z=(z_n)_{n=1}^{\infty}\in c_0$,

$$z=\sum_{n=1}^{\infty} (z_n - z_{n+1}) f_n \qquad (6.14)$$

with convergence in $\|\ \|_{\infty}$. For $N\in\mathbb{N}$,

$$\sum_{n=1}^{N} (z_n - z_{n-1}) f_n = \sum_{n=1}^{N} z_n f_n - \sum_{n=1}^{N} z_{n+1} f_n$$

$$= z_1 f_1 + \sum_{n=2}^{N} z_n f_n - \sum_{n=2}^{N+1} z_n f_{n-1}$$

$$= z_1 f_1 + \sum_{n=2}^{N} z_n (f_n - f_{n-1}) - z_{N+1} f_N$$

$$= z_1 \delta_1 + \sum_{n=2}^{N} z_n \delta_n - z_{N+1} f_N$$

$$= \sum_{n=1}^{N} z_n \delta_n - z_{N+1} f_N . \qquad (6.15)$$

Therefore,

$$\| z - \sum_{n=1}^{N} (z_n - z_{N-1}) f_n \|_{\infty} = \| \sum_{n=N+1}^{\infty} z_n \delta_n + z_{N+1} f_N \|_{\infty}$$

$$\leq \Big\| \sum_{n \geq N+1}^{\infty} z_n \delta_n \Big\|_\infty + |z_{N+1}| \, \|f_N\|_\infty$$

$$= \sup_{n \geq N+1} |z_n| + |z_{N+1}| \to 0 \text{ cuando } N \to \infty,$$

since $\lim_{N \to \infty} |z_N| = 0$. It remains to prove that the representation (6.4) is unique. Suppose that we have $\sum_{n=1}^{\infty} \alpha_n f_n = 0$. Then

$$0 = \sum_{n=1}^{\infty} \alpha_n \Big(\sum_{k=1}^{n} \delta_k \Big) = \sum_{k=1}^{\infty} \Big(\sum_{n \geq k}^{\infty} \alpha_n \Big) \delta_k$$

Since $\{\delta_k\}_{k=1}^{\infty} = \mathcal{E}$ is a basis for $(c_0, \|\,\|_\infty)$ we conclude that for all $k = 1, 2, 3, \dots$, $\sum_{n \geq k}^{\infty} \alpha_n = 0$. Then, for all $k = 1, 2, 3, \dots$

$$\alpha_k = \sum_{n \geq k}^{\infty} \alpha_n - \sum_{n \geq k+1}^{\infty} \alpha_n = 0$$

To find the basis constant of this basis use (6.14) and (6.15) to obtain for all $N \in \mathbb{N}$

$$\|S_N\|_{op} = \sup_{z \neq 0} \frac{\|S_N(z)\|_\infty}{\|z\|_\infty} = \sup_{z \neq 0} \frac{\| \sum_{n=1}^{N} (z_n - z_{n+1}) f_n \|_\infty}{\|z\|_\infty}$$

$$\leq \sup_{z \neq 0} \frac{\| \sum_{n=1}^{N} z_n \delta_n \|_\infty + |z_{n+1}| \, \|f_n\|_\infty}{\|z\|_\infty}$$

$$\leq \sup_{z \neq 0} \frac{\|z\|_\infty + \|z\|_\infty}{\|z\|_\infty} = 2$$

Hence $k_b = \sup_{N \in \mathbb{N}} \|S_N\|_{op} \leq 2$. On the other hand

$$\|S_1\|_{op} \geq \frac{\|S_1(2f_1 - f_2)\|_\infty}{\|2f_1 - f_2\|_\infty} = \frac{\|2f_1\|_\infty}{\|\delta_1 - \delta_2\|_\infty} = 2$$

Hence, $2 \leq k_b \leq \|S_1\|_{op} = 2$ implies $k_b = 2$.

The following is a useful and simple criterion to decide if a given sequence in a Banach space is a Schauder basis.

Proposition 6.12 (*) Let $\beta = \{x_n\}_{n=1}^{\infty}$ be a sequence of elements in a Banach space $(\mathbb{B}, \|\ \|)$. Then β is a Schauder basis if and only if the following three conditions hold:

(i) $x_n \neq 0$ for all $n \in \mathbb{N}$

(ii) $\exists\, C > 0$ such that for every $\{a_n\}_{n=1}^{\infty} \subset \mathbb{F}$, and $m < k$, we have

$$\left\| \sum_{n=1}^{m} a_n x_n \right\| \leq C \left\| \sum_{n=1}^{k} a_n x_n \right\|$$

(iii) $\operatorname{span}\{x_n\}_{n=1}^{\infty}$ is dense in $\mathbb{B}$.

P/ Assume β is a Schauder basis in $(\mathbb{B}, \|\ \|)$. It follows from the uniqueness of the representation $x = \sum_{n=1}^{\infty} \alpha_n(x) x_n$, $x \in \mathbb{B}$, (see (6.4)) that $x_n \neq 0$ and $\operatorname{span}\{x_n\}_{n=1}^{\infty}$ dense in $\mathbb{B}$. Item (ii) is Exercise 6.C.

If (i) and (ii) hold, if $0 = \sum_{n=1}^{\infty} \alpha_n x_n$ (convergence in $\|\ \|$) from (ii), for all $k > 1$

$$\|\alpha_1 x_1\| \leq C \left\| \sum_{n=1}^{k} \alpha_n x_n \right\| \xrightarrow{k \uparrow \infty} 0$$

This implies $\alpha_1 = 0$, since $x_1 \neq 0$. Thus $0 = \sum_{n=2}^{\infty} \alpha_n x_n$ with convergence in norm $\|\ \|$. Repeat the argument for $n = 2$, and continue in this way to show that $\alpha_n = 0$ for all $n = 1, 2, \ldots$. This shows uniqueness of the representation (6.4).

(*) In Albiac-Kalton text book, Proposition 6.12 is called Grunblum's criterion

We need to show that every $x \in \mathbb{B}$ has an expression as in (6.4) with convergence in norm $\| \ \|$. Let S be the elements of $\mathbb{B}$ that can be obtain as convergence series $\sum_{n=1}^{\infty} \alpha_n x_n$ with $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{F}$. It can be proved that S is a vector subspace of $\mathbb{B}$. Using (ii) it can be proved that S is closed in $\mathbb{B}$.

Since $\operatorname{span}\{e_n\}_{n=1}^{\infty} \subset S \subset \mathbb{B}$ and $\overline{\operatorname{span}\{e_n\}_{n=1}^{\infty}} = \mathbb{B}$,

we conclude $S = \bar{S} = \mathbb{B}$.

<u>Exercise 6.F</u> (Difference basis in ℓ^1) Let $\mathcal{E} = \{\delta_n\}_{n=1}^{\infty}$ be the canonical basis of $(\ell^1, \| \ \|_1)$ (See Example 6.10). Define

$$x_1 = \delta_1 \ , \quad x_n = \delta_n - \delta_{n-1} \ , \quad n = 2, 3, \dots$$

Show that $\mathcal{B} = \{x_n\}_{n=1}^{\infty}$ is a monotone basis in $(\ell^1, \| \ \|_\infty)$

S/ Observe that $\|x_1\|_{\ell^1} = 1$ and $\|x_n\|_{\ell^1} = 2$ for all $n \geq 2$. Also, if $\{b_n\}_{n=1}^{M}$ are $\mathbb{F}$-scalars

$$\left\| \sum_{n=1}^{M} b_n x_n \right\|_{\ell^1} = \| b_1 \delta_1 + b_2(\delta_2 - \delta_1) + \dots + b_M(\delta_M - \delta_{M-1}) \|_{\ell^1}$$

$$= \| (b_1 - b_2)\delta_1 + (b_2 - b_3)\delta_2 + \dots + (b_{M-1} - b_M)\delta_{M-1} + b_M \delta_M \|_{\ell^1}$$

$$= \sum_{n=1}^{M-1} |b_n - b_{n+1}| + |b_M| \qquad (6.16)$$

The idea is to use <u>Proposition 6.12</u>. <u>Clearly $x_n \neq 0$</u> for all $n \in \mathbb{N}$ and $\overline{\operatorname{span}\{x_n\}_{n=1}^{\infty}} = \overline{\operatorname{span}\{\delta_n\}_{n=1}^{\infty}} = \ell_1$.

This shows (i) and (iii) of Proposition 6.12. To prove (ii) let $m \in \mathbb{N}$. By (6.16) of $\{b_n\}_{n=1}^{\infty}$ is a sequence of scalars in $\mathbb{F}$,

$$\left\| \sum_{n=1}^{m} b_n x_n \right\|_1 = \sum_{n=1}^{m-1} |b_n - b_{n+1}| + |b_m| \leq$$

$$\leq \sum_{n=1}^{m-1} |b_n - b_{n+1}| + |b_m - b_{m+1}| + |b_{m+1}|$$

$$= \sum_{n=1}^{m} |b_n - b_{n+1}| + |b_{m+1}| \overset{(6.16)}{=} \left\| \sum_{n=1}^{m+1} b_n x_n \right\|_1$$

Consequently, for any $m \in \mathbb{N}$ and $\ell \geq 1$, iterating the above inequality gives

$$\left\| \sum_{n=1}^{m} b_n x_n \right\|_1 \leq \left\| \sum_{n=1}^{m+\ell} b_n x_n \right\|$$

This shows (ii) of Prop. 6.12 and also $\| S_N \|_{op} = 1$. Hence $\mathcal{B}$ is a monotone (Schauder) basis in $(\ell^1, \| \, \|_1)$

Proposition 6.13 Let $\mathcal{B} = \{x_n\}_{n=1}^{\infty}$ be a basis of a Banach space $(\mathbb{B}, \| \, \|)$. Let $\{x_n^+\}_{n=1}^{\infty}$ be the coefficient functionals (see (6.11)). By Thm 6.6, $\{x_n^*\}_{n=1}^{\infty} \subset \mathbb{B}^*$. Then, there exists a constant M such that

$$1 \leq \|x_n\| \, \|x_n^*\|_* \leq M \quad , \quad n = 1, 2, 3, \ldots$$

P/ We have already show in the proof of Thm 6.6 that, there exist $C > 0$ s.t. $|x_n^+(x)| \leq \frac{2C}{\|x_n\|} \|x\|$. Therefore,

$$\|x_n^+\|_* = \sup_{x \neq 0} \frac{|x_n^+(x)|}{\|x\|} \leq \sup_{x \neq 0} \frac{2C \|x\|}{\|x_n\| \, \|x\|} = \frac{2C}{\|x_n\|} .$$

This shows that $\|x_n^*\|_* \|x_n\| \leq 2C = M$. for $n = 1, 2, 3, \ldots$

On the other hand (see (6.12)) $x_n^*(x_n) = 1$. Thus

$$\|x_n^*\|_* \geq \frac{x_n^*(x_n)}{\|x_n\|} = \frac{1}{\|x_n\|},$$

which proves the result. QED.

Exercise 6.G. Let $\beta = \{x_n\}_{n=1}^\infty$ be a basis for a Banach space $(\mathbb{B}, \|\ \|)$. Show

(a) $\inf\limits_{n\geq 1} \|x_n\| > 0 \iff \sup\limits_{n\geq 1} \|x_n^*\|_* < \infty$

(b) $\sup\limits_{n\geq 1} \|x_n\| < \infty \iff \inf\limits_{n\geq 1} \|x_n^*\|_* > 0$.

Def 6.14 A basis $\beta = \{x_n\}_{n=1}^\infty$ in a Banach space $(\mathbb{B}, \|\ \|)$ is called a bounded basis if

$$0 < \inf_{n\geq 1} \|x_n\| \leq \sup_{n\geq 1} \|x_n\| < \infty.$$

The basis β is called a normalized basis if $\|x_n\| = 1$, $n = 1, 2, \ldots$

Observe that if $\beta = \{x_n\}_{n=1}^\infty$ is a basis for the Banach space $(\mathbb{B}, \|\ \|)$, then $\beta' = \{x_n/\|x_n\|\}_{n=1}^\infty$ is a normalized basis of the same Banach space

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