5. REVIEW OF HILBERT SPACE THEORY

Def 5.1. Let $\mathbb{H}$ be a vector space over the field $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$). The map

$$\langle \cdot, \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{F}$$
$$(x, y) \to \langle x, y \rangle$$

is an <u>inner product</u> in $\mathbb{H}$ if

1. $\langle x, x \rangle \geq 0 \ \forall x \in \mathbb{H}$ and if $\langle x, x \rangle = 0$, then $x = 0$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in \mathbb{H}$
3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, y \rangle + \beta \langle y, z \rangle \quad \forall \alpha, \beta \in \mathbb{F},\ x, y \in \mathbb{H}$

A vector space $\mathbb{H}$ with an inner product $\langle , \rangle$ is called a <u>pre-Hilbert space</u>

<u>Exercise 5.A</u> Show that in a pre-Hilbert space $(\mathbb{H}, \langle , \rangle)$, $\langle 0, x \rangle = \langle x, 0 \rangle = 0$ and $\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$ for all $\alpha, \beta \in \mathbb{F}$, $x, y, z \in \mathbb{H}$.

<u>Examples 5.2</u>

1. In $\mathbb{F}^n = \{ (z_1, .., z_n) : z_j \in \mathbb{F}, j = 1, 2, .., n \}$ the expression

$$\langle z, w \rangle = \sum_{j=1}^{n} z_n \overline{w_n} \quad , \quad z = (z_1, .., z_n), \quad w = (w_1, .., w_n)$$

is an inner product

2. Let $\ell^2(\mathbb{N}) = \{ z = (z_j)_{i=1}^{n} : z_j \in \mathbb{F} \text{ and } \sum_{j=1}^{n} |z_j|^2 < \infty \}$.

The expression

$$\langle z, w \rangle = \sum_{j=1}^{\infty} z_j \overline{w_j} \quad , \quad z = (z_j)_{j=1}^{\infty}, \quad w = (w_j)_{j=1}^{\infty}$$

is an inner product. The set $\mathbb{N}$ can be replaced by any ordered finite or countable set, i.e. $\mathbb{Z} = \{0, 1, -1, 2, -2, .. \}$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let

$$L^2(X) = \{f : X \to \mathbb{F} : \int_X |f(x)|^2 d\mu(x) < \infty\}.$$

The expression

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)}\, d\mu(x), \quad f, g \in L^2(X)$$

is an inner product.

<u>Prop 5.3.</u> (Cauchy - Schwarz inequality)

> Let $(\mathbb{H}, \langle\,,\,\rangle)$ be a pre-Hilbert space. For any $x, y \in \mathbb{H}$
>
> $$|\langle x, y\rangle| \le \|x\|\,\|y\|$$
>
> where for $x \in \mathbb{H}$, $\|x\| = \sqrt{\langle x, x\rangle}$ ("norm")

P/ For any $\lambda \in \mathbb{F}$,

$$0 \le \|\lambda x + y\|^2 = \langle \lambda x + y, \lambda x + y\rangle = |\lambda|^2\|x\|^2 + \lambda\langle x, y\rangle +$$
$$+ \bar{\lambda}\langle y, x\rangle + \|y\|^2 = |\lambda|^2\|x\|^2 + 2\,\mathrm{Real}(\lambda\langle x, y\rangle) + \|y\|^2$$

If $x = 0$ the result is clear (see Exercise 3.A). When $x \neq 0$, take $\lambda = -\dfrac{\overline{\langle x, y\rangle}}{\|x\|^2}$ and replace this value in the above inequality:

$$0 \le \frac{|\langle x, y\rangle|^2}{\|x\|^4}\|x\|^2 + 2\,\mathrm{Real}\left(-\frac{\overline{\langle x, y\rangle}}{\|x\|^2}\langle x, y\rangle\right) + \|y\|^2$$

$$= \|y\|^2 + \frac{|\langle x, y\rangle|^2}{\|x\|^2} - 2\frac{|\langle x, y\rangle|^2}{\|x\|^2} = \|y\|^2 - \frac{|\langle x, y\rangle|^2}{\|x\|^2}.$$

This proves the result. QED

<u>Exercise 5.B</u> Let $(\mathbb{H}, \langle\,,\,\rangle)$ be a pre-Hilbert space. Use Cauchy-Schwarz inequality (Prop 3.3) to show that for all $x, y \in \mathbb{H}$, $\|x + y\| \le \|x\| + \|y\|$ and conclude that $\|\ \|$ is a norm in the vector space $\mathbb{H}$.

Exercise 5.C Let $(\mathbb{H}, \langle , \rangle)$ be a pre-Hilbert space with norm $\|x\| = \sqrt{\langle x,x \rangle}$. Thus a normed vector space by Ex 3.B

a) Show the reverse triangle inequality

$$\big|\|x\| - \|y\|\big| \leq \|x-y\| \quad \forall x,y \in \mathbb{H}$$

b) Use a) to show that the map $\| \ \| : \mathbb{H} \to \mathbb{R}$ given by $\|x\| = \sqrt{\langle x,x \rangle}$ is continuos, that is if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{H}$ such that $\lim_{n\to\infty} x_n = x$ (in $(\mathbb{H}, \| \ \|)$)

$$\lim_{n\to\infty} \|x_n\| = \|x\|$$

c) If $\lim_{n\to\infty} x_n = x$ (in $\mathbb{H}, \| \ \|$) and $\lim_{n\to\infty} y_n = y$ (in $(\mathbb{H}, \| \ \|)$), show that $\lim_{n\to\infty} \langle x_n, y_n \rangle = \langle x, y \rangle$. In other words, the map $\langle , \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{F}$ is continuous when $\mathbb{H} \times \mathbb{H}$ has the product topology.

For a pre-Hilbert space $(\mathbb{H}, \langle , \rangle)$ the polarization identity holds. If $\mathbb{F} = \mathbb{C}$

$$\langle x,y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \|x + i^k y\|^2 \quad , \quad x,y \in \mathbb{H}. \qquad (5.1)$$

For $\mathbb{F} = \mathbb{R}$

$$\langle x,y \rangle = \frac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2\right) \quad , \quad x,y \in \mathbb{H} \qquad (5.2)$$

Exercise 3.B shows that any inner product in a vector space $\mathbb{H}$ gives rise to a norm, $\|x\| = \sqrt{\langle x,x \rangle}$. On the other hand, not every norm in a vector space gives rise to an inner product. This is true only if the norm satisfies the parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$, $\forall x,y$. In this case the inner product is defined by (5.1) or (5.2) as the case may be.

Definition 5.4 Let $(\mathbb{H}, \langle , \rangle)$ be a pre-Hilbert space

1. For $x, y \in \mathbb{H}$, x is orthogonal to y, $x \perp y$, if $\langle x, y \rangle = 0$
2. For $E, F \subset \mathbb{H}$, the notation $E \perp F$ means $x \perp y$ for all $x \in E$, $y \in F$
3. For $E \subset \mathbb{H}$, $E^{\perp} = \{y \in \mathbb{H} : y \perp x \text{ for all } x \in E\}$ is the orthogonal subspace of E in $(\mathbb{H}, \langle , \rangle)$.

Exercise 5.D (Pythagoras Theorem) If $\{x_1, \ldots, x_n\} \subset \mathbb{H}$ and $\langle x_i, x_j \rangle = 0$ when $i \neq j$, then $\| \sum_{i=1}^{n} x_i \|^2 = \sum_{i=1}^{n} \|x_i\|^2$.

If $(\mathbb{H}, \langle , \rangle)$ is a pre-Hilbert space, we know by Ex 5.B that $(\mathbb{H}, \| \|)$, where $\|x\| = \sqrt{\langle x, x \rangle}$, is a normed vector space. Thus, we can talk about Cauchy sequences and convergence of sequences in $(\mathbb{H}, \| \|)$.

Definition 5.5 The pre-Hilbert space $(\mathbb{H}, \langle , \rangle)$ is a Hilbert space if the normed vector space $(\mathbb{H}, \| \|)$, with $\|x\| = \sqrt{\langle x, x \rangle}$, is a Banach space.

Historically, the notion of Hilbert space is previous to the one of Banach space. We have prefered to do it the other way around since the notion of Banach space has already been defined in Section 1. Nevertheless, recall that a pre-Hilbert space $(\mathbb{H}, \langle , \rangle)$ is a Hilbert space if every Cauchy sequence $\{x_n\}_{n\geq 1}^{\infty}$ in $(\mathbb{H}, \| \|)$ converges in $(\mathbb{H}, \| \|)$ to an element $x \in \mathbb{H}$.

Let $(X, \mathcal{M}, \mu)$ be a measure space, $X \neq \emptyset$. The map

$$\langle\, , \,\rangle : L^2(X) \times L^2(X) \longrightarrow \mathbb{F}$$

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)}\,d\mu(x) \qquad (5.3)$$

is an inner product on $L^2(X)$ (with the usual assumptions that two functions are equal in $L^2(X)$ if they coincide μ-a.e.). Therefore, $(L^2(X), \langle\, ,\,\rangle)$ is a pre-Hilbert space.

Since

$$\langle f, f\rangle := \int_X |f(x)|^2 d\mu(x) = \|f\|_2^2 \quad , \ f \in L^2(X),$$

theorem 1.3.11 shows that $(L^2(X), \langle\, ,\,\rangle)$ is a Hilbert space, where $\langle\, ,\,\rangle$ is given by (5.3).

In particular $(\mathbb{F}^n, \langle\, ,\,\rangle)$ with the inner product given in example 5.2.1 and $(\ell^2(\mathbb{N}), \langle\, ,\,\rangle)$ with the inner product given in example 5.2.2 are Hilbert spaces.

Exercise 5. E. Let $C([0,1]) = \{ f : [0,1] \to \mathbb{C} : f \text{ continuous} \}$ be the vector space of continuous functions in $[0,1]$.

(a) Show that $C([0,1]) \subset L^2([0,1])$. - Lebesgue measure -

(b) With $\langle f, g\rangle = \int_X f(x)\overline{g(x)}\,d\mu(x)$, $(C([0,1]), \langle\, ,\,\rangle)$ is a pre-Hilbert space. Show that it is not a Hilbert space. (Hint: consider the sequence

$$f_n(x) := \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} \\ [n(x-\frac{1}{2})]^{1/2} & \text{if } \frac{1}{2} < x < a_n = \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } a_n \le x \le 1 \end{cases} \in C([0,1]).)$$

Remark: Compare this result with theorem 1.2.2 where it is proved that $(C[0,1], \|\ \|_\infty)$ is a Banach space.

Prop 5.6. Let $(\mathbb{H}, \langle , \rangle)$ be a pre-Hilbert space. For $y \overset{\neq 0}{\in} \mathbb{H}$ define $T_y : \mathbb{H} \to \mathbb{F}$ by $T_y(x) = \langle x, y \rangle$. Prove that T_y is linear and that

$$\|T_y\| := \sup_{\|x\|=1} |T_y(x)| = \|y\|.$$

P/ Linearity follows from the properties of the inner product.
Taking $x = y/\|y\|$ $(y \neq 0)$,

$$\|T_y\| \geq T_y\left(y/\|y\|\right) = \frac{\langle y, y\rangle}{\|y\|} = \|y\|.$$

For the reverse inequality, by Cauchy-Schwarz inequality

$$\|T_y\| = \sup_{\|x\|=1} |\langle x, y\rangle| \leq \sup_{\|x\|=1} \|x\|\|y\| = \|y\|.$$ QED

If M is a closed subspace of a Hilbert space $(\mathbb{H}, \langle , \rangle)$, it can be proved that $\mathbb{H} = M \oplus M^\perp$, that is

1. $\mathbb{H} = M + M^\perp$ and 2. $M \cap M^\perp = \{0\}$ (5.4)

Recall that $M^\perp$ is always a closed subspace of $\mathbb{H}$, even if M is not closed. On the other hand, the condition of M been closed is necessary for the above result.

For a closed subspace M of a Hilbert space $(\mathbb{H}, \langle , \rangle)$, (5.4) implies that every $x \in \mathbb{H}$ can be written as $x = x_1 + y_1$ with $x_1 \in M$ and $y_1 \in M^\perp$ and this decomposition is unique.
Therefore the maps

$$\mathbb{P}_M : \mathbb{H} \to M \quad \text{given by} \quad \mathbb{P}_M(x) = x_1$$

and

$$\mathbb{P}_{M^\perp} : \mathbb{H} \to M^\perp \quad \text{given by} \quad \mathbb{P}_{M^\perp}(x) = y_1$$

are well defined. These are called <u>orthogonal projections</u> of $\mathbb{H}$ onto the closed subspaces M and $M^\perp$, respectively.

Observe that since $\mathbb{P}_M(x) \in M$ and $\mathbb{P}_{M^\perp}(x) \in M^\perp$, we have $\langle \mathbb{P}_M(x), \mathbb{P}_{M^\perp}(x) \rangle = 0 \ \forall x \in \mathbb{H}$. By Pythagoras Theorem

$$\|\mathbb{P}_M(x)\|^2 + \|\mathbb{P}_{M^\perp}(x)\|^2 = \|x\|, \quad x \in \mathbb{H}.$$

<u>Exercise 5.F</u> Let M be a closed subspace of a Hilbert space $(\mathbb{H}, \langle , \rangle)$. Show that $\mathbb{P}_M : \mathbb{H} \to M^{\subset \mathbb{H}}$ is a linear operator, $\|\mathbb{P}_M\| = 1$, $\mathbb{P}_M \circ \mathbb{P}_M = \mathbb{P}_M$ and $\mathbb{P}_M^* = \mathbb{P}_M$ (self-adjoint).

<u>Remark</u>. If $T \in \mathcal{L}(\mathbb{H}, \mathbb{H})$, the adjoint of T is the unique operator T^* such that

$$\langle x, T^*(y) \rangle = \langle T(x), y \rangle \quad \forall x, y \in \mathbb{H}.$$

<u>Corollary 5.7</u> Let M be a closed subspace of a Hilbert space $(\mathbb{H}, \langle , \rangle)$. If $M \neq \mathbb{H}$, there exists $y \neq 0$, $y \in \mathbb{H}$ such that $y \perp M$.

P/ Since $M \subsetneq \mathbb{H}$, there exists $x \in \mathbb{H} \setminus M$. Take $y = \mathbb{P}_{M^\perp}(x)$. Clearly $y \in M^\perp$ and $y \neq 0$ (otherwise, $x = \mathbb{P}_M(x) + \mathbb{P}_{M^\perp}(x) = \mathbb{P}_M(x)$ would imply $x \in M$, contrary to our choice of x). QED

Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$). The set $\{x_1, \dots, x_n\} \subset V$ is <u>linearly independent</u> (l.i) if when $\sum_{i=1}^{n} \alpha_i x_i = 0$ for some $\alpha_i \in \mathbb{F}$, then $\alpha_i = 0$ for all $i = 1, 2, \dots, n$

A set $S \subset V$ is <u>linearly independent</u> (l.i) if every finite subset of S is l.i.

For a set $S \subset V$, we call $\mathrm{span}(S)$ the set of all finite linear combinations of elements of S, that is

$$\mathrm{span}(S) = \left\{ \sum_{j=1}^{k} \alpha_j x_j : x_j \in S,\ \alpha_j \in \mathbb{F},\ k \in \mathbb{N} \right\}$$

It can be proved that $\mathrm{span}(S)$ is a vector subspace of V and it coincides with the smallest subspace of S that contains S.

Def 5.8. A collection $\{e_i : i \in I\}$ of a pre-Hilbert space $(\mathbb{H}, \langle , \rangle)$ is said to be an <u>orthonormal system</u> if

$$\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} = \delta_{ij} \quad \text{(Kronecker notation)}$$

A subset of S of a normed vector space $(X, \|\ \|)$ is <u>dense</u> in $(X, \|\ \|)$ if $\overline{S} = X$, where $\overline{S}$ indicates the closure of S in the topology induced in X by $\|\ \|$. In other words, for every $x \in X$ and every $\varepsilon > 0$, there exists $y \in S$ such that $\|x - y\| \leq \varepsilon$.

A normed vector space $(X, \|\ \|)$ is said to be <u>separable</u> if it contains a countable set S which is dense in $(X, \|\ \|)$. Recall that $(\mathbb{R}, |\ |)$ is separable, since $\mathbb{Q}$ is dense in $\mathbb{R}$ and countable. Similarly $(\mathbb{C}, |\ |)$ is also separable, since $\mathbb{Q}(i) = \{a + ib : a, b \in \mathbb{Q}\}$ is dense in $(\mathbb{C}, |\ |)$ and countable. In general $(\mathbb{F}^n, \|\ \|_2)$ is separable.

Proposition 5.9. Let $(\mathbb{H}, \langle , \rangle)$ be a separable pre-Hilbert space. Every orthonormal system of $\mathbb{H}$ is countable (finite or infinite)

P/ Let $S = \{e_i : i \in I\}$ be an orthonormal system of $\mathbb{H}$. The goal is to prove that I is a countable set of indices. If $i \neq j$, by Pythagoras theorem, $\|e_i - e_j\|^2 = \|e_i\|^2 + \|e_j\|^2 = 2$. This implies that

$$B_{1/2}(e_i) \cap B_{1/2}(e_j) = \emptyset \qquad (5.4)$$

where $B_{1/2}(x) = \{y \in \mathbb{H} : \|x - y\| < 1/2\}$ is the (open) ball of radius $1/2$ centered at $x \in \mathbb{H}$. In fact, if $x \in B_{1/2}(e_i) \cap B_{1/2}(e_j)$, by the triangle inequality

$$\sqrt{2} = \|e_i - e_j\| \leq \|e_i - x\| + \|x - e_j\| < \tfrac{1}{2} + \tfrac{1}{2} = 1$$

which is a contradiction.

Let $A = \{x_n : n \in J\}$ a dense and countable set of $\mathbb{H}$. Then, for each $e_i \in S$ there exists $x_{n_i} \in A$ such that $x_{n_i} \in B_{1/2}(e_i)$. Since $B_{1/2}(e_i) \cap B_{1/2}(e_j) = \emptyset$ by (5.4), the number of elements of S may not be larger than de number of elements in A. Since A is countable, S is also countable. QED

We will alway consider Hilbert spaces to be separable

If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal system in a Hilbert space (see Def 5.8), for all $x \in \mathbb{H}$, Bessel's inequality holds:

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2 \qquad (5.5)$$

In this case, the partial sums of the series $\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$

form a Cauchy sequence in $\mathbb{H}$. Since $\mathbb{H}$ is a Banach space the series $\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ converges to an element $\tilde{x} \in \mathbb{H}$.

When does $\tilde{x}$ coincide with x?

Def 5.10 A sequence $\{e_n\}_{n=1}^{\infty}$ in a Hilbert space $(\mathbb{H}, \langle , \rangle)$ is said to be TOTAL if $\overline{\operatorname{span}\{e_n\}_{n=1}^{\infty}} = \mathbb{H}$

Warning: Some authors use the term complete sequence instead of total sequence. We have preferred in these notes to use total sequence to avoid confusion with completeness of a norm in a vector space (Def 1.1.3).

Remark: If $\{e_n\}_{n=1}^{\infty}$ is total in $(\mathbb{H}, \langle , \rangle)$, the closed vector subspace $M := \overline{\operatorname{span}\{e_n\}_{n=1}^{\infty}}$ coincides with $\mathbb{H}$. By (5.4) $M^{\perp} = \{0\}$. Hence is total if and only if whenever $x \perp e_n$ for all $n \in \mathbb{N}$, we have $x = 0$.

A key result in the theory of Hilbert spaces is the following

Theorem 5.11 Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in a Hilbert space $(\mathbb{H}, \langle , \rangle)$. The following are equivalent:

(a) $\{e_n\}_{n=1}^{\infty}$ is total in $\mathbb{H}$

(b) $\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n = x$ (convergence of the partial sums in $\| \ \|$) for all $x \in \mathbb{H}$.

(c) (Plancherel identity) For all $x \in \mathbb{H}$

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|x\|^2.$$

<u>Exercise 5.6</u> Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in a Hilbert space $(\mathbb{H}, \langle , \rangle)$. Prove that the following two conditions are equivalent

1. $\{e_n\}_{n=1}^{\infty}$ is total in $\mathbb{H}$
2. (Parseval identity) For all $x, y \in \mathbb{H}$

$$\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle y, e_n \rangle}$$

(Hint: Use theorem 5.11)

S/ 1. ⇒ 2. Since $\{e_n\}_{n=1}^{\infty}$ is <u>total</u> on $\mathbb{H}$ we can use part (b) of Thm 5.11. Then, for $x, y \in \mathbb{H}$

$$\langle x, y \rangle = \left\langle \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n, y \right\rangle = \left\langle \lim_{N\to\infty} \sum_{n=1}^{N} \langle x, e_n \rangle e_n, y \right\rangle$$

$$\underset{\text{of } \langle , \rangle}{\overset{\text{(continuity}}{=\!=}}\ \lim_{N\to\infty} \left\langle \sum_{n=1}^{N} \langle x, e_n \rangle e_n, y \right\rangle \ \overset{\text{linearity of } \langle \cdot, y \rangle}{=\!=\!=}$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \langle x, e_n \rangle \langle e_n, y \rangle = \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle y, e_n \rangle}$$

The last equality holds because the series converges absolutely by Cauchy-Schwarz inequality for $\ell^2(\mathbb{N})$ and Bessel's inequality (5.5):

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle \overline{\langle y, e_n \rangle}| \le \left(\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} |\langle y, e_n \rangle|^2 \right)^{1/2} \le \|x\| \|y\|$$

2. ⇒ 1) Taking $x = y$ in Parseval's identity we deduce Plancherel's identity. By Theorem 5.11, $\{e_n\}_{n=1}^{\infty}$ is total in $\mathbb{H}$.

Def 5.12 A sequence $\{e_n\}_{n=1}^{\infty}$ of a Hilbert space $(\mathbb{H}, \langle , \rangle)$ is an orthonormal basis (o.n.b) for $\mathbb{H}$ if it is an orthonormal system which is total in $\mathbb{H}$

By Theorem 5.11, when $\{e_n\}_{n=1}^{\infty}$ is an o.n.b for $\mathbb{H}$, we have

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \quad \text{(convergence of partial sums in } \mathbb{H})$$

The number $\langle x, e_n \rangle$ is called the n-th coefficient of x with respect to the o.n.b. $\{e_n\}_{n=1}^{\infty}$ (some authors use the name n-th Fourier coefficient). These coefficients are unique.

Exercise 5.H Show that if $\{e_n\}_{n=1}^{\infty}$ is an o.n.b for $\mathbb{H}$ and $x = \sum_{n=1}^{\infty} \alpha_n e_n$, $\alpha_n \in \mathbb{F}$, (in $\mathbb{H}$), then $\alpha_k = \langle x, e_k \rangle$ for all $n = 1, 2, 3, \ldots$

S/ $\langle x, e_k \rangle = \langle \sum_{n=1}^{\infty} \alpha_n e_n, e_k \rangle = \langle \lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n e_n, e_k \rangle$

$= \lim_{N\to\infty} \langle \sum_{n=1}^{N} \alpha_n e_n, e_k \rangle = \lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n \langle e_n, e_k \rangle$

Now $\sum_{n=1}^{N} \alpha_n \langle e_n, e_k \rangle = \left\{ \begin{array}{l} 0 \text{ if } N < k \\ \alpha_k \text{ if } N \geq k \end{array} \right\} \xrightarrow{N \uparrow \infty} \alpha_k.$

Thus $\alpha_k = \langle x, e_k \rangle$. for all $k = 1, 2, \ldots$

Exercise 5.I. In $\ell^2(\mathbb{N})$ with $\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \bar{y}_k$, show that $e_n(k) = \delta_{n,k} = \left\{ \begin{array}{l} 1 \text{ if } k = n \\ 0 \text{ if } k \neq n \end{array} \right\}$ is an orthonormal basis of $\ell^2(\mathbb{N})$

S/ Clearly $\langle e_n, e_m \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$.

If $x = (a_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ we can write $x = \sum_{n=1}^{\infty} a_n e_n$

since $\| x - \sum_{n=1}^{N} a_n e_n \|^2 = \| \sum_{n=N+1}^{\infty} a_n e_n \|^2 = \sum_{n=N+1}^{\infty} |a_n|^2 \to 0$

as $N \to \infty$.

Also, $\langle x, e_k \rangle = \langle (\tilde{a}_n)_{n=1}^{\infty}, e_k \rangle = a_k$. Thus, part (b) of Thm 5.11 holds.

Exercise 5.J In $L^2([a,b])$ show that

$$\{e_n(x) = \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} x} : n \in \mathbb{Z}\}, \quad (T = b-a)$$

is an o.n. system with $\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\,dx$.

S/ $\langle e_n, e_n \rangle = \frac{1}{T}\int_a^b e^{2\pi i \frac{n}{T}x} \cdot e^{-2\pi i \frac{n}{T}x}\,dx = \frac{1}{T}\int_a^b dx = 1$

If $n \neq m$

$$\langle e_n, e_m \rangle = \frac{1}{T}\int_a^b e^{2\pi i \frac{(n-m)}{T}x}\,dx = \frac{1}{T}\left[\frac{e^{2\pi i \frac{(n-m)}{T}x}}{2\pi i \frac{(n-m)}{T}}\right]_a^b$$

$$= \frac{1}{2\pi i(n-m)}\left[e^{2\pi i \frac{(n-m)}{T}b} - e^{2\pi i \frac{(n-m)}{T}a}\right]$$

$$= \frac{1}{2\pi i(n-m)}\left[e^{2\pi i \frac{(n-m)}{T}\cdot(T+a)} - e^{2\pi i \frac{(n-m)}{T}a}\right]$$

$$= \frac{1}{2\pi i(n-m)} e^{2\pi i \frac{n-m}{T}a}\left[e^{2\pi i(n-m)} - 1\right] = 0$$

Remarks. It will be shown that the o.n. system of Ex 5.J is in fact an o.n.b. for $L^2([a,b])$. This requires to study convergence of Fourier series and prove that the trigonometric polynomials, that is, finite linear combinations of exponentials, are dense in $L^2([a,b])$ (recall Def 5.10).

Exercise 5.K Consider the double indexed functions

$$f_{n,k}(x) = e^{2\pi i n x}\chi_{[k,k+1)}(x), \quad x \in \mathbb{R}$$

Show that $\{f_{n,k} : n, k \in \mathbb{Z}\}$ is an o.n.b. for $L^2(\mathbb{R})$ with the inner product given by $\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx$

5/ $\|f_{n,k}\|^2 = \int_k^{k+1} |e^{2\pi i n x}|^2 dx = 1$

If $(n_1, k_1) \neq (n_2, k_2)$

$$\langle f_{n_1,k_1}, f_{n_2,k_2} \rangle = \int_{\mathbb{R}} e^{2\pi i (n_1 - n_2) x} \chi_{[k_1, k_1+1)}(x)\, \chi_{[k_2,k_2+1)}(x)\, dx$$

$$= \left\{ \begin{array}{l} 0 \text{ if } k_1 \neq k_2 \text{ because } [k_1, k_1+1) \cap [k_2, k_2+1) = \emptyset \\ \int_{k_1}^{k_1+1} e^{2\pi i (n_1 - n_2) x} dx = 0 \quad \text{if } k_1 = k_2, \text{ but } n_1 \neq n_2 \end{array} \right\}$$

It remains to show that it is a basis for $L^2(\mathbb{R})$. This will hold if Plancherel identity holds (Thm 5.11).

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} \int_k^{k+1} |f(x)|^2 dx =$$

$$= \sum_{k=-\infty}^{\infty} \|f \cdot \chi_{[k,k+1)}\|^2_{L^2([k,k+1))}$$

Assume that $\{e_n(x) = e^{2\pi i n x} : n \in \mathbb{Z}\}$ is an o.n.b. of $L^2([k,k+1))$.

By Plancherel identity for $L^2([k,k+1])$,

$$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\langle f \cdot \chi_{[k,k+1)}, e_n \rangle_{L^2([k,k+1))}|^2$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left| \int_k^{k+1} f(x) e^{-2\pi i n x} dx \right|^2$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\langle f, e_n \cdot \chi_{[k,k+1)} \rangle_{L^2(\mathbb{R})}|^2$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\langle f, f_{n,k} \rangle_{L^2(\mathbb{R})}|^2$$

Since Plancherel identity holds for $\{f_{n,k} : n, k \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$, we conclude, by Thm 5.11, that $\{f_{n,k}\}_{n,k}$ is total in $L^2(\mathbb{R})$, and, hence, an o.n.b.

If $\{e_k\}_{k=1}^{N}$ is an o.n.b. for a Hilbert space $\mathbb{H}$, we say that $\mathbb{H}$ has dimension N. Otherwise, we say that $\mathbb{H}$ is infinity dimensional.

In an infinite dimensional Hilbert space $(\mathbb{H}, <,>)$, which is separable, the Gram-Schmidt ortonormalization process applied to a countable dense set of $\mathbb{H}$, produces an orthonormal basis for $\mathbb{H}$. Hence, every separable Hilbert space has an o.n.b. In fact, any separable Hilbert space is isometrically isomorphic to $\ell^2(\mathbb{N})$.

Theorem 5.13 (Riesz-Fisher Theorem)

Let $\{e_n\}_{n=1}^{\infty}$ be an o.n.b of a separable Hilbert space $(\mathbb{H}, <,>)$. Then $U: \mathbb{H} \to \ell^2(\mathbb{N})$ given by

$$U(x) = \{ <x, e_n> \}_{n=1}^{\infty}$$

is an isometric isomorphism.

The isometry part follows from Plancherel identity (Theorem 5.11 (c)):

$$\| U(x) \|^2_{\ell^2(\mathbb{N})} = \sum_{n=1}^{\infty} |<x, e_n>|^2 = \|x\|^2$$

-U is clearly lineal

Exercise L Show that the map U given in Theorem 5.13 is an inyective and biyective map.