4. Linear Operators on Norm Vector Spaces

Let $(X_1, \|\ \|_1)$ and $(X_2, \|\ \|_2)$ be two normed vector spaces over $\mathbb{F}$. A map $T: X_1 \to X_2$ is a <u>linear operator</u> if for every $x, y \in X_1$ and every $\alpha, \beta \in \mathbb{F}$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

The set of linear operators from X_1 onto X_2 will be denoted by $\mathcal{L}(X_1, X_2)$.

<u>Def 4.1</u> Let $(X_1, \|\ \|_1)$ and $(X_2, \|\ \|_2)$ be two normed vector spaces over $\mathbb{F}$ and $T \in \mathcal{L}(X_1, X_2)$. The "norm" of T is defined by

$$\|T\| := \sup_{x \neq 0, x \in X_1} \frac{\|Tx\|_2}{\|x\|_1}$$

The operator T is said to be <u>bounded</u> if $\|T\| < \infty$

The set of boundet operators from X_1 onto X_2 will be denoted by $\mathcal{B}(X_1, X_2)$.

From the definition, for every $x \in X_1$, $\|T\| \geq \frac{\|Tx\|_2}{\|x\|_1}$ i.e.

$$\|Tx\|_2 \leq \|T\| \|x\|_1 \qquad (4.1)$$

and $\|T\| = \inf_{x \in X, x \neq 0} \{ c > 0 : \|Tx\|_2 \leq c\|x\| \}$.

Also, since $x/\|x\|_1 = y$ has norm 1 in X_1 $(x \neq 0)$,

$$\|T\| = \sup_{\|y\|_1 = 1} \|Ty\|_2 \qquad (4.2)$$

Exercise 4.A The set $\mathcal{B}(X_1, X_2)$ is a vector space with the operations $T_1 + T_2 (x) = T_1(x) + T_2(x)$ and $(\alpha . T)(x) = \alpha . T(x)$. Show that $\|\ \|$ is a norm in $\mathcal{B}(X_1, X_2)$.

S/ Clearly $\|T\| \geq 0$. If $T = 0$, $\|T\| = 0$. Moreover if $\|T\|=0$, by definition, $\|Tx\|_2=0$ for all $x\in X_1$, $x\neq 0$. Hence $T(x)=0$ for all $x\in X_1$.

The property $\|\alpha T\|=|\alpha|\,\|T\|$ for $\alpha\in\mathbb{F}$ is clear from Def 4.1.

Let $T_1, T_2 \in \mathcal{B}(X_1,X_2)$. Then, by (4.1)

$$\|T_1+T_2\| = \sup_{\|y\|_1=1} \|(T_1+T_2)(y)\|_2 \le \sup_{\|y\|_1} \left(\|T_1(y)\|_2 + \|T_2(y)\|_2\right)$$

$$\le \sup_{\|y\|_1=1} \|T_1(y)\|_2 + \sup_{\|y\|_1=1} \|T_2(y)\| = \|T_1\| + \|T_2\|$$

A linear operator $T\in\mathcal{L}(X_1,X_2)$ is continuous at $x_1\in X_1$ if $\forall \varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that if $\|x-x_1\|_1\le\delta$ then $\|T(x)-T(x_1)\|\le\varepsilon$. $T\in\mathcal{L}(X_1,X_2)$ is continuous in X_1 if it is continuous for every $x_1\in X_1$.

Theorem 4.2. $T\in\mathcal{L}(X_1,X_2)$, $(X_1,\|\ \|_1)$ and $(X_2,\|\ \|_2)$ normed vector spaces. The following conditions are equivalent

(a) T is bounded

(b) T is continuous

(c) T is continuous at one point of X_1

P/ (a) $\Rightarrow$ (b) By (4.1), if $x,y\in X_1$, $\|T(x)-T(y)\|_2 \le \|T\|\,\|x-y\|_1$. Hence, if $\varepsilon>0$, choose $\delta=\varepsilon/\|T\|$, to conclude that if $\|x-y\|_1<\delta$, then $\|T(x)-T(y)\|_2 \le \|T\|\,\frac{\varepsilon}{\|T\|}=\varepsilon$.

(b) $\Rightarrow$ (c) Trivial

(c) $\Rightarrow$ (a) Suppose that T is continuous at $x_0\in X_1$. For each $\varepsilon=1$, there exists $\delta>0$ such that if $\|x\|_1\le\delta$, then $\|T(x_0+x)-T(x_0)\|_2\le 1$. Since T is linear, we have

$\|T(x)\|_2 \le 1$ when $\|x\|_1 \le \delta$. Hence, if $\|y\|_1 = 1$, let $x = \delta y$, so that $\|x\|_1 = \delta$, we deduce $\|T(x)\|_2 \le 1 \iff \|T(\delta y)\|_2 \le 1 \iff \|T(y)\|_2 \le 1/\delta$. By (4.2)

$$\|T\| = \sup_{\|y\|_1 = 1} \|Ty\|_2 \le \frac{1}{\delta},$$

which shows that T is bounded. QED.

Theorem 4.3 Let $(X, \|\ \|_1)$ be a normed vector space and $(\mathbb{B}, \|\ \|_2)$ be a Banach space. The space $\mathcal{B}(X, \mathbb{B})$ of bounded linear operators from X into $\mathbb{B}$ is a Banach space.

P/ Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{B}(X, \mathbb{B})$. For $x \in X$, and $n, m \in \mathbb{N}$, (4.1) gives

$$\|T_n(x) - T_m(x)\|_2 = \|(T_n - T_m)(x)\|_2 \le \|T_n - T_m\| \, \|x\|_1. \qquad (4.3)$$

Since $\|T_n - T_m\| \to 0$ as $n, m \to \infty$, this implies that for each $x \in X$, $\{T_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{B}$. Since $(\mathbb{B}, \|\ \|_2)$ is a Banach space, there exists $y \in \mathbb{B}$ such that $\lim_{n\to\infty} T_n(x) = y$. Define $T : X \to \mathbb{B}$ by $T(x) = y = \lim_{n\to\infty} T_n(x)$.

Exercise 4.B Prove that T is linear

S/ If $T(x_1) = \lim_{n\to\infty} T_n(x_1)$ and $T(x_2) = \lim_{n\to\infty} T_n(x_2)$

$$T(x_1 + x_2) = \lim_{n\to\infty} T_n(x_1 + x_2) = \lim_{n\to\infty} T_n(x_1) + T_n(x_2) = T(x_1) + T(x_2)$$

Similarly, if $\alpha \in \mathbb{F}$ and $T(x) = \lim_{n\to\infty} T_n(x)$, then

$$T(\alpha \cdot x) = \lim_{n\to\infty} T_n(\alpha \cdot x) = \alpha \lim_{n\to\infty} T_n(x) = \alpha T(x) \quad \blacksquare$$

We need to prove that T is bounded. Since $\{T_n\}_{n=1}^{\infty}$ is

Cauchy in $\mathcal{B}(X,\mathbb{B})$, there exists $N\in\mathbb{N}$ such that $\|T_n-T_m\|\le 1$ for all $n,m\ge N$. From (4.3), $\|T_n(x)-T_m(x)\|_2\le\|x\|_1$ for all $x\in X$. Taking the limit when $n\to\infty$ and recalling that $\|\ \|_2$ is continuous.(Ex 1.1.A), we obtain
$\|T(x)-T_m(x)\|_2\le\|x\|_1$ $\forall x\in X$ and all $m\ge N$. Hence

$$\|T(x)\|\le\|T(x)-T_N(x)\|+\|T_N(x)\|\le(1+\|T_N\|)\|x\|_1$$

which shows that

$$\|T\|\le 1+\|T_N\|<\infty.$$

It remains to prove that $\lim_{n\to\infty} T_n = T$ in $(\mathcal{B}(X,\mathbb{B}),\|\ \|)$. Let $\varepsilon>0$. Since $\{T_n\}_{n=1}^{\infty}$ is Cauchy in $(\mathcal{B}(X,\mathbb{B}),\|\ \|)$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that $\|T_n-T_m\|\le\varepsilon$ when $n,m\ge N$. From (4.3), for every $x\in X$,

$$\|T_n(x)-T_m(x)\|_2\le\varepsilon\|x\|_1$$

Taking limits when $n\to\infty$ (recall that $\|\ \|_2$ is continuous)

$$\|T(x)-T_m(x)\|_2\le\varepsilon\|x\|_1$$

for all $x\in X$ and all $m\ge N$. Hence

$$\|T-T_m\| = \sup_{\substack{x\in X\\ x\neq 0}}\frac{\|(T-T_m)(x)\|_2}{\|x\|_2}\le\varepsilon \qquad \forall m\ge N(\varepsilon)$$

Q.E.D.

This shows the result.

Def 4.4.

Let $(X,\|\ \|)$ be a normed vector space over $\mathbb{F}$. The dual space of X, denoted by X^*, is the set $\mathcal{B}(X,\mathbb{F})$ of all bounded linear functionals of X.

If $x^*\in X^*=\mathcal{B}(X,F)$,

$$\|x^*\|_* = \sup_{\substack{x\neq 0\\ x\in X}}\frac{|x^*(x)|}{\|x\|_1} = \sup_{\|y\|=1}|x^*(y)|$$

makes X^* into a normed vector space (Ex. 4.A) which is a Banach space by Thm 4.3 since $(\mathbb{F}, |\ |)$ is a Banach space.

Given a normed space $(X, \|\ \|)$, it is very useful to identify its dual X^*. This question is solved in the case of the spaces $L^p(X)$, $1 \le p < \infty$, when $(X, \mathcal{M}, \mu)$ is a σ-finite measure space.

Let $1 \le p < \infty$, and $g \in L^q(X)$, where $\frac{1}{q} + \frac{1}{p} = 1$.

Define
$$\ell_g(f) = \int_X f(x)\overline{g(x)}\, d\mu(x), \quad f \in L^p(X) \qquad (4.4)$$

Exercise 4.C Show that $\ell_g \in (L^p(X))^* = \mathcal{B}(L^p(X), \mathbb{F})$

S/ The map ℓ is linear by the linearity of the integral. To prove that is bounded, let $f \in L^p(X)$, $\|f\|_p = 1$. By Hölder's inequality

$$|\ell_g(f)| \le \int_X |f(x)||g(x)|\, d\mu(x) \le \|f\|_p \|g\|_q = \|g\|_q$$

Hence, by (4.2), $\|\ell_g\|_* \le \|g\|_q < \infty$.

Identifying g with ℓ_g, this shows that $L^q(X) \subset (L^p(X))^*$

An important results in Functional Analysis is the following

Thm 4.5 (Duality for Lebesgue spaces)

Let $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Then

$$(L^p(X))^* = L^q(X)$$

in the following sense. For every $\ell \in (L^p(X))^*$, there is a unique $g \in L^q(X)$ such that

$$\ell(f) = \int_X f(x)\overline{g(x)}\, d\mu(x), \quad \forall f \in L^p(X)$$

Moreover

$$\|\ell\|_* = \|g\|_q$$

The non-trivial prove of this result uses the Radon-Nykodim theorem, which requires that $(X, \mathcal{M}, \mu)$ be a σ-finite measure space.

Theorem 4.5 shows that $(L^1(X))^* = L^\infty(X)$. However $(L^\infty(X))^*$ contains $L^1(X)$, but it is larger.

Proposition 4.6 Let $(X, \|\ \|)$ be a normed vector space over $\mathbb{F}$. Denote by B_1^* the closed unit ball of the Banach space $(X^*, \|\ \|_*)$. For every $x \in X$,

$$\|x\| = \sup_{x^* \in B_1^*} |x^*(x)|$$

therefore, the map for $x \in X$, the map $f_x : X^* \to \mathbb{F}$ given by $f_x(x^*) = x^*(x)$ is a bounded linear functional on X^* with $\|f_x\|_{op} = \|x\|$.

P/ By a corollary of the Hahn-Banach Theorem (see Corollary to Theorem 3.3 in Rudin's Book (Functional Analysis)), there exist $y^* \in B_1^*$ such that $y^*(x) = \|x\|$. Therefore

$$\sup_{x^* \in B_1^*} |x^*(x)| \geq |y^*(x)| = \|x\|$$

On the other hand, for every $x^* \in B_1^*$, $1 \geq \|x^*\|_* = \sup_{y \in X} |x^*(y)|/\|y\|$ $\geq \frac{|x^*(x)|}{\|x\|} \Rightarrow |x^*(x)| \leq \|x\|$ for all $x^* \in B_1^*$. Thus

$$\sup_{x^* \in B_1^*} |x^*(x)| \leq \|x\|. \quad \blacksquare$$

4.1. MAIN THEOREMS OF FUNCTIONAL ANALYSIS

4.1.1 (Normed Space version of the Hahn - Banach Theorem)

Let y^* be a bounded linear functional on a subspace Y of a normed vector space $(X, \| \|)$. Then, there is $x^* \in X^*$ such that $\|x^*\| = \|y^*\|$ and $x^*/Y = y^*$

Warning The Hahn-Banach Theorem does not talks about uniqueness of the extension (unless Y is dense in X). Notice that Y need not be closed.

4.1.2 (Separation of points from closed subspaces)

Let Y be a closed subspace of a normed vector space $(X, \| \|)$. Suppose that $x \in X \setminus Y$. Then, there exists $x^* \in X^*$ such that $\|x^*\|_* = 1$, $x^*(x) = \inf\{\|x-y\| : y \in Y\}$ and $x^*(y) = 0$ $\forall y \in Y$.

Corollary 4.1.3. Let $(X, \| \|)$ be a normed vector space and $x \in X$, $x \neq 0$. There exists $x^* \in X^*$ such that $\|x^*\|_* = 1$ and $x^*(x) = \|x\|$.

P/ Choose $Y = \{0\}$ and apply 4.1.2.

Corollary 4.1.4. Let $(X, \| \|)$ be a normed vector space. For every $x \in X$ we have $\|x\| = \sup_{x^* \in B_1^*} |x^*(x)|$, where $B_1^* = \{x^* \in B^* : \|x^*\|_* \leq 1\}$.

The proof is given in Proposition 4.6.

A map $f : (X, \mathcal{E}_1) \to (Y, \mathcal{E}_2)$ -topological spaces- is open if for all $V \in \mathcal{E}_1$, $f(V) \in \mathcal{E}_2$. The map f is continuous in X if for all $U \in \mathcal{E}_2$, $f^{-1}(U) = \{x \in X : f(x) \in U\} \in \mathcal{E}_1$.

4.1.5 (The open mapping theorem)

Let $(X, \|\|_1)$ and $(Y, \|\|_2)$ be Banach spaces and $T: X \to Y$ bounded linear operator.

(i) If $B_\delta(0;Y) := \{y \in Y : \|y\|_2 < \delta\} \subset \overline{T(B_1(0;X))}$ for some $\delta > 0$, then T is an open map

(ii) If is onto, the hypothesis in (i) hods. That is, every bounded linear operator from a Banach space onto a Banach space is open

Corollary 4.1.6. Let X, Y be Banach spaces an T a continuosus linear operator from X onto Y that is also one to one. Then $T^{-1}: Y \to X$ is a continuous linear operator

P/ Just observe that $T^{-1}: Y \to X$ continuous if and only if $T: X \to Y$ is open

4.1.6. (Closed graph theorem)

Let X, Y be Banach spaces. Suppose that $T \in \mathcal{L}(X, Y)$ such that if $\{x_n\} \subset X$ satisfies $\lim_{n\to\infty} x_n = x \in \mathbb{R}$ and $y = \lim_{n\to\infty} T(x_n)$ exists in Y, it follows that $y = Tx$. Then $T \in \mathcal{B}(X, Y)$ i.e. $T: X \to Y$ is continuous.

4.1.7 (Uniform boundedness principle)

Let $\{T_\gamma\}_{\gamma \in \Gamma}$ be a family of bounded linear operators from a Banach space $(X, \|\|_1)$ into a normed vector space $(Y, \|\|_2)$. Suppose $\sup_{\gamma \in \Gamma} \|T_\gamma x\|_2 < \infty$ for each $x \in X$. Then $\sup_{\gamma \in \Gamma} \|T_\gamma\|_{op} < \infty$

4.1.8 (Banach-Steinhaus theorem)

Let $(T_n)_{n=1}^{\infty}$ be a sequence of continuous linear operator from $(X, \|\|_1)$ Banach into $(Y, \|\|_2)$ normed s.v. $T(x) = \lim_{n\to\infty} T_n(x)$ exists for all $x \in X$. Then T is continuous (and linear)