3. L^p SPACES (LEBESGUE SPACES)

Let $(X, \mathcal{M}, \mu)$ be a (σ-finite) measure space: $X \neq \emptyset$ is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ a measure

Def 1.3.1 Let $X \neq \emptyset$ be a set an $\mathcal{M}$ a family of subsets of X. $\mathcal{M}$ is a σ-algebra in X if

1. $\emptyset \in \mathcal{M}$ 2. If $A \in \mathcal{M}$, $A^c \in \mathcal{M}$ 3. If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$

Def 1.3.2. If $\mathcal{M}$ is a σ-algebra in $X \neq \emptyset$, $\mu : \mathcal{M} \to [0, \infty]$ is a (positive) measure if

1) $\mu(\emptyset) = 0$ 2. If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ disjoint, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

The elements of $\mathcal{M}$ are called measurable sets

Def 1.3.3. A (positive) measure μ on a σ-algebra $\mathcal{M}$ is said to be σ-finite if $\exists \{E_k\}_{k=1}^{\infty} \subset \mathcal{M}$ such that $\mu(E_k) < \infty$ and $X = \bigcup_{k=1}^{\infty} E_k$. The measure μ is finite if $\mu(X) < \infty$. If $\mu(X) = 1$, $(X, \mathcal{M}, \mu)$ is called a probability space.

A) If J is a finite or countable set (i.e. $J = \mathbb{N}$ or $\mathbb{Z}$), $\mathcal{M} = \mathcal{P}(J)$ the collection of all subsets of J is a σ-algebra and $\mu(A) = \mathrm{Card}(A)$ for all $A \subset J$ is a σ-finite measure space (finite if J is finite)

B) If X is a topological space (in particular if $(X, \| \, \|)$ is a normed vector space) with open sets given by the collection $\mathcal{T}$, we called $\mathcal{B}(X, \mathcal{T})$ the σ-algebra generated by $\mathcal{T}$, that is $\mathcal{B}(X, \mathcal{T}) = \{ \bigcap_{\mathcal{M} \supset \mathcal{T}} \mathcal{M} : \mathcal{M}\ \sigma\text{-algebra of } X \}$

In particular, any subset of X obtained as countable union or intersection of open or closed sets belongs to $\mathcal{B}(X)$.
$\mathcal{B}(X, \tau)$ is called the Borel σ-algebra of (X, τ).

Recall that if $(\mathbb{B}, \|\ \|)$ is a normed vector space, the (open) balls $\{B_r(x) : x \in \mathbb{B}, r > 0\}$ generate the topology of $\mathbb{B}$. In particular, the topology of $\mathbb{F}$ is generated by the "intervals" and $\mathcal{B}(\mathbb{F})$ coincide with the σ-algebra generated by the "open intervals".

In $\mathbb{R}$ (or $\mathbb{R}^n$, $n \geq 2$) we will always use the Lebesgue measure studied in any first year course in measure theory. This can also be defined on $\mathbb{C}$ since $\mathbb{C}$ can be considered as $\mathbb{R} \times \mathbb{R}$ for this purposes. Similarly, Lebesgue measure can be defined in $\mathbb{F}^n$.

Def 1.3.4. Let $\mathcal{M}$ be a σ-algebra on $X \neq \emptyset$. A function $f : X \to \mathbb{R}$ is said to be measurable (or $\mathcal{M}$-measurable) if for all $I \subset \mathbb{R}$ interval, $f^{-1}(I) = \{x \in X : f(x) \in I\} \in \mathcal{M}$.

C) In the above situation, $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable if the functions $Re(f) : X \to \mathbb{R}$ and $Im(f) : X \to \mathbb{R}$ are $\mathcal{M}$-measurable.

D) $\mathcal{M}(X; \mathbb{F})$ will denote the collection of all measurable functions from X into $\mathbb{F} = (\mathbb{R}$ or $\mathbb{C})$.

A function $s : X \to \mathbb{C}$ is called <u>SIMPLE</u> if there exists $E_1, \dots, E_n \in \mathcal{M}$ and $c_1, \dots, c_n \in \mathbb{C}$ such that

$$s(x) = \sum_{j=1}^{n} c_j \chi_{E_j}(x)$$

where $\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$ is the characteristic function of the set $E \in \mathcal{M}$.

Def 1.3.5. For a simple function $s = \sum_{j=1}^{n} c_j \cdot \chi_{E_j}$ on $(X, \mathcal{M}, \mu)$ the integral of s is defined as

$$\int_X s \, d\mu = \sum_{j=1}^{n} c_j \, \mu(E_j)$$

(Remark: If for some j, $\mu(E_j) = \infty$ we decide $\int_X s \, d\mu = \infty$.)

Def 1.3.6. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $f : X \to \mathbb{C}$ a measurable function.

1. If $f(x) \geq 0 \ \forall x \in X$, define

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu : s \text{ simple and } 0 \leq s \leq f \right\}$$

2. If $f : X \to \mathbb{R}$, write $f = f^+ - f^-$ where

$$f^+(x) = \max\{f(x), 0\} = \frac{f(x) + |f(x)|}{2}$$

$$f^-(x) = \frac{f(x) - |f(x)|}{2} = -\min\{f(x), 0\}.$$

If $\int_X f^+ d\mu < \infty$ and $\int_X f^- d\mu < \infty$, define

$$\int_X f \, d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$$

3. If $f : X \to \mathbb{C}$, define

$$\int_X f \, d\mu = \int_X \operatorname{Re} f(x) \, d\mu + i \int_X \operatorname{Img} f(x) \, d\mu.$$

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Define the collection of all <u>integrable</u> functions as

$$\mathcal{L}(X,\mathcal{M},\mu) = \mathcal{L}(X) := \{ f : X \to \mathbb{F} : f \text{ measurable} \\ \text{and } \int_X |f| \, d\mu < \infty \}$$

The set $\mathcal{L}(X)$ is a vector space with

$$(f+g)(x) = f(x) + g(x) \quad , \quad f, g \in \mathcal{L}(X), \; x \in X$$

$$(\lambda f)(x) = \lambda \cdot f(x) \quad , \quad \lambda \in \mathbb{F}, \; f \in \mathcal{L}(X), \; x \in X.$$

In $\mathcal{L}(X)$ it seems natural to define

$$\|f\|_1 = \int_X |f| \, d\mu$$

It is easy to see that

$$\|\lambda f\|_1 = \int_X |\lambda f| \, d\mu = |\lambda| \, \|f\|_1 \qquad \forall \lambda \in \mathbb{F}, \; f \in \mathcal{L}(X)$$

and if $f, g \in \mathcal{L}(X)$

$$\|f+g\|_1 = \int_X |f+g| \, d\mu \le \int_X (|f| + |g|) \, d\mu =$$

$$= \int_X |f| \, d\mu + \int_X |g| \, d\mu = \|f\|_1 + \|g\|_1$$

Thus, 1) and 2) of the definition of norm hold (Def 1.1.1). It is also clear that $\|f\|_1 \ge 0$ for all $f \in \mathcal{L}(X)$ and that if $f(x) = 0 \; \forall x \in X$, then $\|f\|_1 = 0$. But from $\|f\|_1 = \int |f| \, d\mu = 0$ one can only deduce that:

$$\mu(\{x \in X : f(x) \ne 0\}) = 0$$

This is commonly written as $f(x) = 0$ a.e. (almost everywhere) $x \in X$. For example, if $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$ on $[0,1]$, $\int_0^1 |f| = 0$, but $f \ne 0$.

To avoid this problem we identify functions that are equal except for a set of measure zero. That is, in $\mathcal{L}(X)$ we say that $f \sim g$ if $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$ ($\Leftrightarrow$ $f(x) = g(x)$ μ-a.e. $x \in X$). This is an equivalent relation in $\mathcal{L}(X)$

Let us consider now the quotient

$$L^1(X, \mathcal{M}, \mu) = L^1(X) := \mathcal{L}(X)/_\sim = \{[f] : f: X \to \mathbb{C} \text{ measurable and } \int_X |f| d\mu < \infty\} \quad (3.1)$$

and for $[f] \in L^1(X)$, define

$$\|[f]\|_1 := \int_X |f| d\mu. \quad (3.2)$$

This is well defined since two functions that coincide a.e. $x \in X$ have the same value for the integral.

By an abuse of notation, we will write $\|f\|_1$ instead of $\|[f]\|_1$, but we must remember that $\|f\|_1 = 0$ means that $f(x) = 0$ a.e. $x \in X$.

Lema 1.3.7

The space $L^1(X, \mathcal{M}, \mu) = L^1(X)$ defined in (3.1) is a normed vector space with the norm given by (3.2).

For $1 < p < \infty$, let

$$L^p(X, \mathcal{M}, \mu) = L^p(X) := \{f: X \to \mathbb{C} : f \text{ measurable and } \int_X |f|^p d\mu < \infty\} \quad (3.3)$$

where, as before, we identify two functions that are equal a.e. $x \in X$.

Lema 1.3.8

For $1 < p < \infty$, the space $L^p(X, \mathcal{M}, \mu) = L^p(X)$ defined in (3.3) is a vector space.

P/ If $f \in L^p(X)$ and $\lambda \in \mathbb{F}$, $\int_X |\lambda f|^p d\mu = |\lambda|^p \int_X |f|^p d\mu < \infty$ so that $\lambda f \in L^p(X)$. It remains to show that if $f, g \in L^p(X)$, then $f+g \in L^p(X)$. To prove this imitate the solution of Exercise 1.2.C which is the case $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$ and $\mu(A) = \text{Card}(A)$ for $A \in \mathcal{M}$.

It is natural to consider the following map

$$\|f\|_p := \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}, \quad f \in L^p(X) \qquad (3.4)$$

with the usual identification of functions a.e. All the properties of a norm (see Def 1.1.1) are easy to prove, except the triangle inequality. To do this, we need Hölder's inequality.

Prop 1.3.9 (Hölder's inequality for L^p spaces)
Let $(X, \mathcal{M}, \mu)$ be a (σ-finite) measure space and $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ (conjugate exponents). If $f \in L^p(X)$ and $g \in L^q(X)$, the function $f \cdot g \in L^1(X)$ and

$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q .$$

P/ If $\|f\|_p = 0$ or $\|g\|_q = 0$ the inequality is trivial. Assume $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$. For each $x \in X$, use lemma 1.2.6 ($ab \leq \frac{a^p}{p} + \frac{b^p}{q}$, $a, b \geq 0$) with $a = \frac{|f(x)|}{\|f\|_p}$ and $b = \frac{|g(x)|}{\|g\|_q}$ to obtain

$$\frac{|f(x)|}{\|f\|_p} \cdot \frac{|g(x)|}{\|g\|_q} \leq \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}, \quad x \in X.$$

Integrating over X,

$$\int_X \frac{|f(x)|}{\|f\|_p} \cdot \frac{|g(x)|}{\|g\|_q} d\mu(x) \leq \frac{1}{p} \int_X \frac{|f(x)|^p}{\|f\|_p^p} d\mu(x) + \frac{1}{q} \int_X \frac{|g(x)|^q}{\|g\|_q^q} d\mu(x)$$

$$= \frac{1}{p} + \frac{1}{q} = 1.$$

Thus show the result:

$$\|fg\|_1 = \int_X |f(x)g(x)|\,d\mu(x) \leq \|f\|_p \|g\|_q.$$

QED.

Theorem 1.3.10 Let $1 < p < \infty$. The space $L^p(X)$ given in (3.3) is a normed vector space with $\|\ \|_p$ given by (3.4)

P/ As mentioned before Prop 1.3.9, it is enough to show that if $f, g \in L^p(\mu)$,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \qquad (3.5)$$

which is known as Minkowski's inequality.

By the triangle inequality in $(\mathbb{R}, |\ |)$,

$$\|f+g\|_p^p = \int_X |f(x)+g(x)|\,|f(x)+g(x)|^{p-1}\,d\mu(x)$$

$$\leq \int_X |f(x)|\,|f(x)+g(x)|^{p-1}\,d\mu(x) + \int_X |g(x)|\,|f(x)+g(x)|^{p-1}\,d\mu(x)$$

By Hölder's inequality (Prop 1.3.9),

$$\|f+g\|_p^p \leq \left(\int_X |f(x)|^p\,d\mu(x)\right)^{1/p}\left(\int_X |f(x)+g(x)|^{q(p-1)}\,d\mu(x)\right)^{1/q}$$

$$+ \left(\int_X |g(x)|^p\,d\mu(x)\right)^{1/p}\left(\int_X |f(x)+g(x)|^{q(p-1)}\,d\mu(x)\right)^{1/q} \qquad (3.6)$$

Observe that $\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = pq \iff q(p-1) = p$. Thus

$$\left(\int_X |f(x)+g(x)|^{q(p-1)}\,d\mu(x)\right)^{1/q} = \|f+g\|_p^{p/q}$$

Replacing this equality in (3.6) we obtain

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)\,\|f+g\|_p^{p/q}$$

Since $p(1-\frac{1}{q}) = 1$, we conclude that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Observe that if $\|f+g\|_p = 0$, (3.5) is trivial.

QED

σ-finite

Theorem 1.3.11. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \le p < \infty$. Then $(L^p(X), \|\ \|_p)$ is a Banach space, where $\|f\|_p$ is defined in (3.4).

P/ Let $\{f_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $L^p(X)$. With $\epsilon = \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^k}, \ldots$ we can find a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ $(n_{k+1} \ge n_k)$ of $\{f_n\}_{n\ge 1}^{\infty}$ such that

$$\| f_{n_{k+1}} - f_{n_k} \|_p < \frac{1}{2^k}, \quad k \in \mathbb{N}. \qquad (3.7)$$

For $m = 1, 2, 3, \ldots$ consider

$$g_m(x) := \sum_{k=1}^{m} |f_{n_{k+1}}(x) - f_{n_k}(x)| \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

By the triangle inequality in $(L^p, \|\ \|_p)$ (Prop 1.3.10) and (3.7) we obtain

$$\|g_m\|_p \le \sum_{k=1}^{m} \| f_{n_{k+1}} - f_{n_k} \|_p \le \sum_{k=1}^{m} \frac{1}{2^k} \le \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \qquad (3.8)$$

By Fatou's lemma and (3.8) we obtain

$$\int_X g(x)^p \, d\mu(x) = \int_X \lim_{m\to\infty} g_m(x)^p \, d\mu(x) \le \liminf_{m\to\infty} \int_X g_m(x)^p \, d\mu(x)$$

$$= \liminf_{m\to\infty} \|g_m\|^p \le 1$$

We have shown that $g \in L^p(X)$. In particular $g(x) < \infty$ a.e. $x \in X$ (argue by contradiction). Let $A(g) = \{x \in X : g(x) = \infty\}$ and $A(f_{n_1}) = \{x \in X : f_{n_1}(x) = \infty\}$. We know that $\mu(A(g)) = 0$ and $\mu(A(f_{n_1})) = 0$. For $x \in X \setminus (A(g) \cup A(f_{n_1}))$,

$$f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$$

is a numerical series that converges absolutely since

$$|f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| = |f_{n_1}(x)| + |g(x)| < \infty$$

Define
$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$$
when $x \in X - (A(g) \cup A(f_{n_1}))$ and $f(x) = 0$ if $x \in A(g) \cup A(f_{n_1})$.

Since
$$f_{n_1}(x) + \sum_{k=1}^{m-1} f_{n_{k+1}}(x) - f_{n_k}(x) = f_{n_m}(x)$$
it follows that
$$\lim_{m \to \infty} f_{n_m}(x) = f(x) \quad \text{a.e. } x \in X. \qquad (3.4)$$

It remains to prove

I. $f \in L^p(X)$ II. $\{f_n\}_{n=1}^{\infty} \to f$ in $(L^p(X), \|\,\|_p)$.

Let $\varepsilon > 0$. Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(L^p(X), \|\,\|_p)$ there is $N = N(\varepsilon) \in \mathbb{N}$ such that $\|f_n - f_m\|_p < \varepsilon$ for all $n, m \geq N$. In particular, by (3.4) and Fatous' lemma

$$\int_X |f(x) - f_n(x)|^p d\mu(x) = \int_X \lim_{m\to\infty} |f_{n_m}(x) - f_n(x)|^p d\mu(x)$$
$$\leq \liminf_{m \to \infty} \int_X |f_{n_m}(x) - f_n(x)|^p d\mu(x)$$
$$= \liminf_{m \to \infty} \|f_{n_m} - f_n\|^p < \varepsilon^p \quad \text{when } n \geq N \quad (3.5)$$

In particular, $f - f_N \in L^p(X)$ and $f = (f - f_N) + f_N \in L^p(X)$ since $L^p(X)$ is a vector space. This proves I.

The proof of II is precisely (3.5) since it reads $\|f - f_n\|_p < \varepsilon$ for all $n \geq N$. QED.

In the proof of Theorem 1.3.11 we have used a fundamental result in integration theory, which we state for the reader's convenience

Lemma 1.3.12 (Fatou's Lemma)

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions in X such that $f_n(x) \geq 0$ a.e. $x \in X$, then

$$\int_X \left(\liminf_{n\to\infty} f_n(x)\right) d\mu(x) \leq \liminf_{n\to\infty} \int_X f_n(x)\, d\mu(x)$$

Other important results to know from Lebesgue integration theory are the Monotone Convergence Theorem (MCT) and the Lebesgue Dominate Convergence Theorem (LDC). We encourage the reader to review these results.

In the proof of Theorem 1.3.11 we have obtained the following result which is worth to emphasize (see (3.6))

Prop 1.3.13.

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $1 \leq p < \infty$. If $\{f_n\}_{n=1}^{\infty} \subset L^p(X)$ and converge to $f \in L^p(X)$ in $\| \ \|_p$, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \quad \text{a.e. } x \in X$$

We now discuss the case $p=\infty$. Consider

$$L^\infty(X,\mathcal{M},\mu)=L^\infty(X)=\{f: X\to\mathbb{C} \ : f \text{ measurable an } \exists\ 0\le C<\infty \text{ s.t. } |f(x)|\le C \text{ a.e. } x\in X\}, \tag{3.6}$$

where, as before, we identify two functions that are equal a.e. $x\in X$.

For $f\in L^\infty(X)$, define

$$\|f\|_\infty=\inf\{C: |f(x)|\le C \text{ a.e. } x\in X\}. \tag{3.7}$$
$$=\operatorname*{ess\,sup}_{x\in X}|f(x)|.$$

<u>Exercise 1.3.A.</u> Show that $(L^\infty(X), \|\,\|_\infty)$ is a normed vector space

S/ For the triangle inequality, observe that if $f\in L^\infty(X)$, $|f(x)|\le \|f\|_\infty$ a.e. $x\in X$. Then if $f,g\in L^\infty(X)$,

$$|f(x)+g(x)|\le|f(x)|+|g(x)|\le\|f\|_p+\|g\|_p$$

a.e. $x\in X$ (since the union of two sets of measure zero has measure zero). By definition (3.6), $f+g\in L^\infty(X)$ and by (3.7)

$$\|f+g\|_\infty\le\|f\|_\infty+\|g\|_\infty.$$

Observe that if $f\in L^1(X)$ and $g\in L^\infty(X)$

$$\|fg\|_1=\int_X|f(x)||g(x)|\,d\mu(x)\le\|g\|_\infty\int_X|f(x)|\,d\mu(x)$$
$$=\|f\|_1\|g\|_\infty$$

which extends Hölder's inequality (Prop 1.3.9) to the case $p=1$ and $q=\infty$.

Exercise 1.3.B. Prove that if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a normed vector space $(\mathbb{B}, \| \ \|)$, then $\{x_n\}_{n=1}^{\infty}$ is bounded, that is, there exists $M < \infty$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Exercise 1.3.C Suppose that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(L^{\infty}(X), \| \ \|_{\infty})$. Show that there exists a measurable set $A \subset X$ with $\mu(A^c) = 0$, such that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{\infty}$$

$\forall n, m \in \mathbb{N}$ and all $x \in A$.

S/ By def. of $\|f_n - f_m\|_{\infty}$ there exists $N_{n,m}$ measurable with $\mu(N_{n,m}) = 0$ such that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{\infty} \qquad (3.8)$$

for all $x \in N_{n,m}^c$. Let $A = \bigcap_{n,m \in \mathbb{N}} N_{n,m}^c$. Since $A^c = \bigcup_{n,m} N_{n,m}$, and $\mu(N_{n,m}) = 0$, $\mu(A^c) = 0$. For all $x \in A$, (3.8) holds for all $n, m \in \mathbb{N}$

Proposition 1.3.14. Let $(X, \mathcal{M}, \mu)$ be a (σ-finite) measure space. Then $(L^{\infty}(X), \| \ \|_{\infty})$ as defined in (3.6) and (3.7) is a Banach space.

Exercise 1.3.D. Prove Proposition 1.3.14

S/ Let A be the set of Ex 1.3.C. Given $\varepsilon > 0$, $\exists\, N = N(\varepsilon) \in \mathbb{N}$ such that $\|f_n - f_m\|_{\infty} < \varepsilon$ for all $n, m \geq N$.

By Exer. 1.3.C, $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m \geq N = N(\varepsilon)$ and all $x \in A$. Let $\tilde{f}_n = f_n|_A$. Then

$$|\tilde{f}_n(x) - \tilde{f}_m(x)| < \varepsilon \qquad \forall n, m \in \mathbb{N}.$$

This shows that $\{\tilde{f}_n(x)\}_{n\geq 1}^{\infty}$ is Cauchy in $\mathbb{F}$ uniformly on A. Since $(\mathbb{F}, |\ |)$ is complete, there exists

$$\lim_{m\to\infty} \tilde{f}_m(x) := f(x) \quad , x\in A$$

Moreover, if $n > N(\varepsilon) \in \mathbb{N}$

$$|\tilde{f}_n(x) - f(x)| = \lim_{m\to\infty} |\tilde{f}_n(x) - \tilde{f}_m(x)| < \varepsilon$$

Choose $f(x) = 0$ for all $x \in A^c$. Since $\mu(A^c) = 0$,

$$\|f_n - f\|_\infty = \sup_{x\in X} |f_n(x) - f(x)| < \varepsilon$$

By Ex. 1.3.B, there is $M < \infty$ such that $\|f_n\| \leq M$ for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$ there exist M_n with $\mu(M_n) = 0$ such that $|f_n(x)| \leq M$ $\forall n \in \mathbb{N}$ and all $x \in M_n^c$. Let $B = \bigcap_{n\in\mathbb{N}} M_n^c$. Then

$$\mu(B^c) = \mu(\bigcup_n M_n) = 0$$

Redefine $\tilde{f}_n(x) = \begin{cases} f_n(x) & \text{if } x \in A\cap B \\ 0 & \text{if } x \in A^c \cup B^c \end{cases}$. We now have that $f(x) = \lim_{n\to\infty} \tilde{f}_n(x)$, $x \in A \cap B$ also satisfies $\|f\|_\infty < \infty$.

In general, there are no inclusions among the spaces $L^p(X)$

Exercise 1.3.E Let $1 \le p < q < \infty$. Show that if $f(x) = x^{-1/q} \chi_{(0,1)}(x)$, then $f \in L^p(\mathbb{R}, dx)$, but $f \notin L^q(\mathbb{R}, dx)$

S/ $\displaystyle\int_0^1 x^{-p/q} dx = \left[\frac{x^{-\frac{p}{q}+1}}{1 - p/q} \right]_0^1 = \frac{1}{1 - p/q}$ (since $\frac{p}{q} < 1$)

and
$$\int_0^1 x^{-q/q} dx = \int_0^1 \frac{1}{x} dx = \infty .$$

Exercise 1.3.D Let $1 \le p < q < \infty$. Show that if $g(x) = x^{-1/p} \chi_{(1,\infty)}(x)$, then $g \in L^q(\mathbb{R}, dx)$, but $g \notin L^p(\mathbb{R}, dx)$

S/ $\displaystyle\int_1^\infty x^{-q/p} dx = \left[\frac{x^{-\frac{q}{p}+1}}{-\frac{q}{p} + 1} \right]_1^\infty = \frac{1}{\frac{q}{p} - 1}$ (since $\frac{q}{p} > 1$)

and $\displaystyle\int_1^\infty x^{-p/p} dx = \int_1^\infty \frac{1}{x} dx = \infty .$

The situation is different if $\mu(X) < \infty$.

Prop 1.3.15. Let (X, M, μ) be a measure space with $\mu(X) < \infty$ and $1 \le p < q \le \infty$. Show that
$$\|f\|_p \le \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q .$$
Hence, $L^q(X) \subset L^p(X)$ if $\mu(X) < \infty$.

P/ If $q = \infty$,
$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} \le \|f\|_\infty \left(\int_X 1 \, d\mu(x) \right)^{1/p} = \|f\|_\infty \, \mu(X)^{1/p}$$

If $q < \infty$ use Hölder's inequality with exponents $p_1 = \frac{q}{p} > 1$ and $q_1 = \frac{q}{q-p}$ which are conjugate exponents:
$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{p}{q} + \frac{q-p}{q} = 1 .$$

Thus

$$\|f\|_p^p = \int_X |f(x)|^p \cdot 1 \, d\mu(x) \leq \left(\int_X |f(x)|^q d\mu(x)\right)^{p/q} \left(\int_X 1 \, d\mu(x)\right)^{\frac{q-p}{q}}$$

$$= \|f\|_q^p \, \mu(X)^{1-\frac{p}{q}}$$

This implies

$$\|f\|_p \leq \|f\|_q \, \mu(X)^{\frac{1}{p}-\frac{1}{q}} \,.$$

QED

When $X = \mathbb{N}$ (or any finite or countable set) with the σ-algebra of $\mathcal{P}(\mathbb{N})$ and μ the counting measure, Exer 1.2.B shows that if $1 \leq p < q \leq \infty$

$$L^p(\mathbb{N}) = \ell^p \subset \ell^q = L^q(\mathbb{N})$$

and the inclusions are the opposite as the ones given in Prop 1.3.15.

<u>Exercise 1.3.E</u> Let $f \in L^\infty(X)$ with $\operatorname{supp} f = E$ and $\mu(E) < \infty$. Show that $f \in L^p(X)$ for all $p < \infty$ and

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty \qquad (\mu(E) = 0 \text{ trivial})$$

S/ $\|f\|_p = \left(\int_E |f(x)|^p d\mu(x)\right)^{1/p} \leq \|f\|_\infty \, \mu(E)^{1/p}$. Thus

$$\limsup_{p \to \infty} \|f\|_p \leq \|f\|_\infty \lim_{p \to \infty} \mu(E)^{1/p} = \|f\|_\infty$$

On the other hand, for $\varepsilon > 0$, $\mu(\{x \in X : |f(x)| \geq \|f\|_\infty - \varepsilon\}) \geq \delta$ for some $\delta > 0$. Then

$$\int_X |f(x)|^p d\mu(x) \geq \int_{\{x \in X : |f(x)| \geq \|f\|_\infty - \varepsilon\}} |f(x)|^p d\mu(x)$$

$$\geq (\|f\|_\infty - \varepsilon)^p \mu(\{x \in X : |f(x)| \geq \|f\|_\infty - \varepsilon\})$$

$$\geq \delta(\|f\|_\infty - \varepsilon)^p .$$

Thus,

$$\liminf_{p\to\infty} \|f\|_p \geq (\|f\|_\infty - \varepsilon) \lim_{p\to\infty} \delta^{1/p} = (\|f\|_\infty - \varepsilon).$$

Since ε is arbitrary, we have

$$\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty$$

Thus, $\lim\limits_{p\to\infty} \|f\|_p$ exists an equals $\|f\|_\infty$. ∎