4.2. EXAMPLES OF BANACH SPACES

In any first year real analysis course it is proved that the absolute value is a norm in $\mathbb{R}$ and that $(\mathbb{R}, |\ |)$ is complete. Thus, $(\mathbb{R}, |\ |)$ is a Banach space. In the other hand, $(\mathbb{Q}, |\ |)$ is not a Banach space.

<u>Exercise</u> 1.2.A. Observe first that if $a, b \in \mathbb{R}^+ \cup \{0\}$,

$$\sqrt{ab} \leq \frac{a+b}{2} \quad (\Leftrightarrow \ 4ab \leq (a+b)^2 = a^2+b^2+2ab \ \Leftrightarrow \ 0 \leq a^2+b^2-2ab = (a-b)^2)$$

(i) Show that if $x_i, y_i \in \mathbb{R}$, $i = 1, 2, .., n$, then

$$\left| \sum_{i=1}^n x_i y_i \right| \leq \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^n |y_i|^2 \right)^{1/2}$$

(Cauchy-Schwartz inequality)

(ii) In $\mathbb{R}^n = \{x = (x_1, .., x_n) : x_j \in \mathbb{R}, j = 1, .., n\}$ show that $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)$ is a norm.

S/ (i) Let $A := \sum_{i=1}^n |x_i|^2$ and $B = \sum_{i=1}^n |y_i|^2$. Use the observation with the numbers $\frac{|x_i|^2}{A} > 0$ and $\frac{|y_i|^2}{B} > 0$ to obtain

$$\sqrt{\frac{|x_i|^2}{A} \cdot \frac{|y_i|^2}{B}} \leq \frac{1}{2} \left[\frac{|x_i|^2}{A} + \frac{|y_i|^2}{B} \right]$$

$$\Leftrightarrow \quad \frac{|x_i|}{\sqrt{A}} \cdot \frac{|y_i|}{\sqrt{B}} \leq \frac{1}{2} \left[\frac{|x_i|^2}{A} + \frac{|y_i|^2}{B} \right]$$

Summing in i,

$$\frac{1}{\sqrt{A}} \frac{1}{\sqrt{B}} \sum_{i=1}^n |x_i||y_i| \leq \frac{1}{2} \left(\sum_{i=1}^n \frac{|x_i|^2}{A} + \sum_{i=1}^n \frac{|y_i|^2}{B} \right) = \frac{1}{2}(1+1) = 1$$

(ii) Enough to prove $\|x+y\|_2 \le \|x\|_2 + \|y\|_2$ since the other two properties of the norm hold trivially. We have,

$$\|x+y\|_2^2 = \sum_{i=1}^n |x_i+y_i|^2 \le \sum_{i=1}^n (|x_i|+|y_i|)^2 =$$

$$= \sum_{i=1}^n |x_i|^2 + \sum_{i=1}^n |y_i|^2 + 2\sum_{i=1}^n |x_i||y_i| \overset{(i)}{\le}$$

$$= \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2 = (\|x\|_2+\|y\|_2)^2 .$$

Proposition 1.2.1.
$(\mathbb{R}^n, \|\ \|_2)$ is a Banach space

P/ We need to prove that $\|\ \|_2$ is complete in $\mathbb{R}^n$. Let $\{x_k\}_{k\ge1}^\infty$ be a Cauchy sequence in $\mathbb{R}^n$. Write $x_k = (x_1^{(k)}, \dots, x_n^{(k)})$.

For each $j = 1, 2, \dots, n$,

$$|x_j^{(k)} - x_j^{(\ell)}| \le \left(\sum_{i=1}^n |x_i^{(k)} - x_i^{(\ell)}|^2\right)^{1/2} = \|x_k - x_\ell\|$$

Thus, for each $j = 1, 2, \dots, n$, $\{x_j^{(k)}\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Since $(\mathbb{R}, |\ \ |)$ is complete, there exists $x_j \in \mathbb{R}$ s.t. $\lim_{k\to\infty} x_j^{(k)} = x_j$.

Given $\varepsilon > 0$, choose $N_j = N_j(\varepsilon) \in \mathbb{N}$ such that $|x_j^{(k)} - x_j| < \frac{\varepsilon}{\sqrt{n}}$ for all $k \ge N_j$, $j = 1, 2, \dots, n$. Let now $k \ge N := \max\{N_1, \dots, N_n\}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then

$$\|x_k - x\|_2^2 \le \sum_{j=1}^n |x_j^{(k)} - x_j|^2 < \sum_{j=1}^n \frac{\varepsilon^2}{n} = \varepsilon^2$$

This shows that $\lim_{k\to\infty} x_k = x$ in $(\mathbb{R}^n, \|\ \|_2)$ and, hence, $(\mathbb{R}^n, \|\ \|_2)$ is complete. Q.E.D

The vector space $\mathbb{C} = \{z = a+ib : a, b \in \mathbb{R}\}$ of complex numbers with $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2+b^2}$ is a normed vector space (use exercise 1.2A with $n=2$) and also complete (same proof as in Prop 1.2.1).

Again, as in Prop. 1.2.1, $\mathbb{C}^n = \{z = (z_1, .., z_n) : z_j \in \mathbb{C}, j = 1, 2, -, n\}$ with $\|z\|_2 = (\sum_{i=1}^{n} |z_i|^2)^{1/2}$ is a Banach space

---

Let K be a compact Hausdorff topological space. Define $C(K)$ the set of all functions $f : K \to \mathbb{F} (= \mathbb{R} \text{ or } \mathbb{C})$ which are continuous in K. For $f \in C(K)$ define

$$\|f\|_\infty = \sup_{t \in K} |f(t)|$$

It is easy to see that $\| \ \|_\infty$ is a norm in $C(K)$, known as the supremum norm.

Theorem 1.2.2 The vector space $C(K)$ defined above is a Banach space with the supremum norm.

P/ Enough to prove completeness. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(C(K), \| \ \|_\infty)$. For each $t \in K$,

$$|f_n(t) - f_m(t)| \le \sup_{s \in K} |f_n(s) - f_m(s)| = \|f_n - f_m\|_\infty .$$

Thus, for each $t \in K$, $\{f_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathbb{F}, | \ |)$. Since $(\mathbb{F}, | \ |)$ is complete, for each $t \in K$, $\lim_{n\to\infty} f_n(t)$ exists. Define $f : K \to \mathbb{F}$ by

$$f(t) = \lim_{n\to\infty} f_n(t) \qquad (2.1)$$

While the function f is well-defined, we still don't know if it is continuous. Recall from a real analysis course that

(i) $\{f_n\}_{n\geq 1}^{\infty}$ converges uniformly to f (functions defined from K onto $\mathbb{F}$) if $\forall\, \varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that for all $n\geq N$, $\|f_n-f\|_{\infty}<\varepsilon$

(ii) If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions that converge uniformly to f, then f is also continuous

In view of (ii) it is enough to show that $\{f_n\}_{n=1}^{\infty}$ converge uniformly to the function f defined in (2.1). Choose $\varepsilon>0$. Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence with $\|\ \|_{\infty}$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that for all $n,m\geq N$ and all $t\in K$

$$|f_n(t)-f_m(t)|<\varepsilon$$

Taking the limit when $m\to\infty$, we obtain that for all $n\geq N$ and all $t\in K$, $|f_n(t)-f(t)|<\varepsilon$. This means that $\|f_n-f\|_{\infty}<\varepsilon$ as wanted. QED.

Prop. 1.2.3. A closed subspace V of a Banach space $(\mathbb{B},\|\ \|)$ is also a Banach space with the same norm.

<u>Exercise</u> 1.2.B. Prove Prop 1.2.3.

S/ We need to prove that V is complete with $\|\ \|$. Let $\{y_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(V,\|\ \|)$. Since $V\subset\mathbb{B}$, $\{y_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in

$(\mathbb{B}, \|\ \|)$. Since $(\mathbb{B}, \|\ \|)$ is complete, there exists $y \in \mathbb{B}$ such that $\lim_{n\to\infty} y_n = y$. Since V is closed, $y \in V$. QED

Let us study now infinite dimensional vector spaces of sequences. Let $1 \leq p < \infty$, and $x = (x_n)_{n=1}^{\infty}$, $x_n \in \mathbb{F}$, define:

$$\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}. \qquad (2.1)$$

The set

$$\ell^p = \ell^p(\mathbb{N}) := \{x = (x_n)_{n=1}^{\infty} : \|x\|_p < \infty\}$$

is a vector space with the operation $x + y = (x_n + y_n)_{n=1}^{\infty}$ and $\lambda x = (\lambda x_n)_{n=1}^{\infty}$, $\lambda \in \mathbb{F}$.

It is clear that if $x \in \ell^p$ and $\lambda \in \mathbb{F}$, then $\|\lambda x\|_p = |\lambda|\|x\| < \infty$.

Exercise 1.2.C. Observe that for $a, b \in \mathbb{R}^+ \cup \{0\}$ and $1 \leq p < \infty$, $(a+b)^p \leq (2\max(a,b))^p = 2^p \max(a^p, b^p) \leq 2^p(a^p + b^p)$ to show that if $x, y \in \ell^p$, then $x + y \in \ell^p$

S/ Let $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$. By the inequality given in the statement, for all $n = 1, 2, 3, \ldots$

$$|x_n + y_n|^p \leq (|x_n| + |y_n|)^p \leq 2^p(|x_n|^p + |y_n|^p)$$

Suming over $n \in \mathbb{N}$

$$\|x+y\|_p^p = \sum_{n=1}^{\infty} |x_n + y_n|^p \leq 2^p\left(\sum_{n=1}^{\infty} |x_n|^p + \sum_{n=1}^{\infty} |y_n|^p\right)$$

$$= 2^p(\|x\|_p^p + \|y\|_p^p) < \infty.$$

Prop 1.2.4. The map $\| \ \|_1 : \ell^1 \to \mathbb{R}^+ \cup \{0\}$ given in (2.1) makes ℓ^1 into a normed vector space.

P/ Properties 1) and 2) of the norm are easy to show. For the triangle inequality, let $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$ be two elements of ℓ^1. Then

$$\|x+y\|_1 = \sum_{n=1}^{\infty} |x_n + y_n| \le \sum_{n=1}^{\infty} |x_n| + |y_n| = \|x\|_1 + \|y\|_1 .$$ QED.

The case $1 < p < \infty$ requires more work. Observe that Exercise 1.2.C gives $\|x+y\|_p \le 2\left(\|x\|_p^p + \|y\|_p^p\right)^{1/p}$, but this is not the triangle inequality for $\| \ \|_p$.

Lema 1.2.5. Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ (p and q are called conjugate exponents). For $a, b \in \mathbb{R}^+ \cup \{0\}$

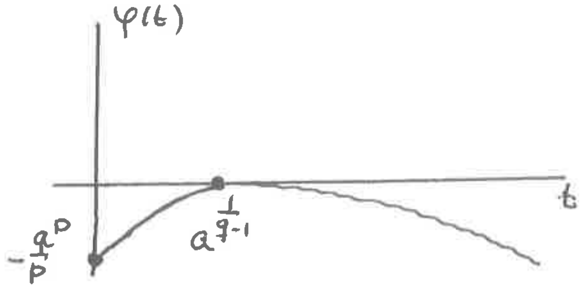
$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

P/ If $a = 0$ or $b = 0$, the inequality is obviously true. For $a > 0$ define $\varphi(t) = at - \frac{1}{p}a^p - \frac{1}{q}t^q$. We want to prove that $\varphi(t) \le 0$ for all $t > 0$. We have $\varphi(0) = -\frac{1}{p}a^p < 0$.

Moreover, $\varphi'(t) = a - t^{q-1} = 0 \Leftrightarrow t^{q-1} = a \Leftrightarrow t = a^{\frac{1}{q-1}}$.

Since $\varphi''(t) = -(q-1)t^{q-2}$, $\varphi''(a^{\frac{1}{q-1}}) = -(q-1)\, a^{\frac{q-2}{q-1}} < 0$

which shows that φ has a local maximum at $t = a^{\frac{1}{q-1}}$ and its value at this point is



$$\varphi(a^{\frac{1}{q-1}}) = a \cdot a^{\frac{1}{q-1}} - \frac{1}{p}a^p - \frac{1}{q}a^{\frac{q}{q-1}} =$$
$$= (1-\tfrac{1}{q})\, a^{\frac{q}{q-1}} - \frac{1}{p}a^p$$

Notice that: $\frac{1}{p}+\frac{1}{q}=1 \iff \frac{1}{p} = 1-\frac{1}{q} = \frac{q-1}{q} \iff$
$p = \frac{q}{q-1}$. Thus

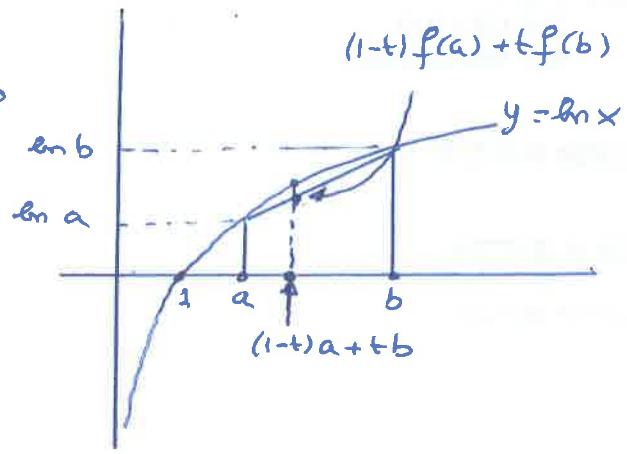
$$\varphi(a^{\frac{1}{q-1}}) = \frac{1}{p}a^p - \frac{1}{p}a^p = 0$$

Moreover, $\lim_{t\to\infty} \varphi(t) = -\frac{1}{p}a^p + \lim_{t\to\infty} t^q(a\,t^{1-q} - \frac{1}{q}) = +\infty$.
Hence $\varphi(t) \le 0 \quad \forall t>0$ as wanted. QED

(Second proof of Prop 1.2.5).

The function $f(x) = \ln x$, $0<x<\infty$ is concave since

$$f''(x) = -\frac{1}{x^2} < 0$$



This implies, that if $0 \le t \le 1$ since $a \le (1-t)a + tb \le b$,

$$(1-t)f(a) + tf(b) \le f((1-t)a + tb)$$

and $\quad (1-t)\ln a + t\ln b \le \ln((1-t)a + tb) \qquad (2.2)$

From (2.2) with $t = \frac{1}{q}$, $1-t = 1-\frac{1}{q} = \frac{1}{p}$ and a^p, b^p instead of a and b we obtain

$$\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q \le \ln(\tfrac{1}{p}a^p + \tfrac{1}{q}b^q)$$

The left hand side of this inequality coincides with $\ln ab$.
Hence $\quad \ln(ab) \le \ln(\frac{1}{p}a^p + \frac{1}{q}b^q)$

and the result follows by taking exp on both sides of this inequality. QED

Prop. 1.2.6. (Hölder's inequality for ℓ^p spaces)

Let $1<p,q<\infty$ be such that $\frac{1}{p}+\frac{1}{q}=1$. If $x=(x_n)_{n=1}^{\infty}\in\ell^p$ and $y=(y_n)_{n=1}^{\infty}\in\ell^q$, the sequence $x\cdot y:=(x_ny_n)_{n=1}^{\infty}$ belongs to ℓ^1 and

$$\|x\cdot y\|_1\leq\|x\|_p\|y\|_q.$$

P/ If $\|x\|_p=0$ or $\|y\|_q=0$ the inequality is trivial. Assume $\|x\|_p\neq 0$ and $\|y\|_q\neq 0$. For each $n=1,2,3,\dots$, Lemma 1.2.6 applied to $a=\frac{|x_n|}{\|x\|_p}$ and $b=\frac{|y_n|}{\|y\|_q}$ gives

$$\frac{x_n\cdot y_n}{\|x\|_p\|y\|_q}\leq\frac{1}{p}\frac{|x_n|^p}{\|x\|_p^p}+\frac{1}{q}\frac{|y_n|^q}{\|y\|_q^q}$$

Suming over $n\in\mathbb{N}$

$$\sum_{n=1}^{\infty}\frac{x_n\cdot y_n}{\|x\|_p\|y\|_q}\leq\frac{1}{p}\sum_{n=1}^{\infty}\frac{|x_n|^p}{\|x\|^p}+\frac{1}{q}\sum_{n=1}^{\infty}\frac{|y_n|^q}{\|y\|^q}$$

$$=\frac{1}{p}+\frac{1}{q}=1.$$

Thus,

$$\|x\cdot y\|_1=\sum_{n=1}^{\infty}|x_n\cdot y_n|\leq\|x\|_p\|y\|_q.$$

QED.

Prop 1.2.7. Let $1<p<\infty$. The map $\|\ \|_p:\ell^p\to\mathbb{R}^+\cup\{0\}$ given in (2.1) is a norm in ℓ^p

P/ Enough to prove the triangle inequality since 1) and 2) of the definition of norm hold trivially. Let $x=(x_n)_{n=1}^{\infty}$ and $y=(y_n)_{n=1}^{\infty}$ be two elements of ℓ^p.

Let $z = (|x_n + y_n|^{p-1})_{n=1}^{\infty}$. Then, with $z_n = |x_n + y_n|^{p-1}$

$$\|x+y\|_p^p = \sum_{n=1}^{\infty} |x_n + y_n|^p = \sum_{n=1}^{\infty} |x_n + y_n||z_n|$$
$$\leq \sum_{n=1}^{\infty} |x_n||z_n| + \sum_{n=1}^{\infty} |y_n||z_n|$$

By Hölder's inequality (Prop 1.2.6)

$$\|x+y\|_p^p \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |z_n|^q\right)^{1/q} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |z_n|^q\right)^{1/q}.$$

Observe that $\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = pq \iff q(p-1) = p$. Thus,

$$\sum_{n=1}^{\infty} |z_n|^q = \sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} = \sum_{n=1}^{\infty} |x_n + y_n|^p = \|x+y\|_p^p.$$

Hence

$$\|x+y\|_p^p \leq (\|x\|_p + \|y\|_p) \|x+y\|_p^{p/q}$$

$$\iff \|x+y\|_p^{p(1-\frac{1}{q})} \leq \|x\|_p + \|y\|_p$$

$$\iff \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Usually called Minkowski's inequality for ℓ^p. QED.

For our exposition we have chosen the index set as $\mathbb{N}$, but all the results we have proved for ℓ^p spaces hold when the index set is any countable ordered set.

Define

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$$

and

$$\ell^\infty = \ell^\infty(\mathbb{N}) = \{x = (x_n)_{n=1}^{\infty} : \|x\|_\infty < \infty\}$$

It is easy to see that $(\ell^\infty, \| \|_\infty)$ is a normed vector space.

Observe that if $x=(x_n)_{n=1}^\infty \in \ell^1$ and $y=(y_n)_{n=1}^\infty \in \ell^\infty$, then

$$\|x \cdot y\|_1 = \sum_{n=1}^\infty |x_n y_n| \leq \|y\|_\infty \sum_{n=1}^\infty |x_n| = \|x\|_1 \|y\|_\infty$$

which is the version of Hölder's inequality for $p=1$ and $q=\infty$.

Theorem 1.2.8

> For $1 \leq p \leq \infty$, $(\ell^p, \| \|_p)$ is a Banach space

P/ We already know (Prop 1.2.7 for $1<p<\infty$, Prop 1.2.4 for $p=1$, and comments before thm 1.2.8 for $p=\infty$) that $(\ell^p, \| \|_p)$ is a normed vector space. It remains to prove that $(\ell^p, \| \|_p)$ is complete. We will do the case $1 \leq p < \infty$ and leave the easier case $p=\infty$ as an exercise.

Let $\{x^k\}_{k=1}^\infty$ be a Cauchy sequence in $(\ell^p, \| \|_p)$. Write

$$x^k = (x_1^k, x_2^k, \ldots, x_n^k, \ldots) \quad , \quad x_n^k \in \mathbb{F}$$

For $n \in \mathbb{N}$

$$|x_n^k - x_n^\ell| \leq \left(\sum_{m=1}^\infty |x_m^k - x_m^\ell|^p\right)^{1/p} = \|x^k - x^\ell\|_p$$

This shows that for each $n \in \mathbb{N}$, the sequence $\{x_n^k\}_{k=1}^\infty \subset \mathbb{F}$ ($=\mathbb{R}$ or $\mathbb{C}$) is a Cauchy sequence. Since $\mathbb{F}$ is complete, there exists $x_n \in \mathbb{F}$ such that $\lim_{k\to\infty} x_n^k = x_n$ (in $\mathbb{F}$)

Define

$$x = (x_1, x_2, \ldots, x_n, \ldots). \tag{2.3}$$

This is our candidate to be the limit of x^k in $(\ell^p, \| \ \|_p)$ when $k \to \infty$.

We need to prove two things.

$$\begin{array}{l} x^1 = (x_1^1, x_2^1, x_3^1, x_4^1 \ldots \\ x^2 = (x_1^2, x_2^2, x_3^2, x_4^2 \ldots \\ x^3 = (x_1^3, x_2^3, x_3^3, x_4^3, \ldots \\ \quad\ \vdots \qquad \downarrow \quad\ \downarrow \quad\ \downarrow \quad\ \downarrow \\ \qquad\ \ x_1 \quad x_2 \quad x_3 \quad x_4 \end{array}$$

I) The element x defined in (2.3) belongs to ℓ^p

II) $\lim_{k\to\infty} x^k = x$ in $\| \ \|_p$.

To do this, we need to pass to a subsequence of $\{x_n^k\}_{k,n}$ as follows. Since $(x_1^k)_{k\geq 1}^{\infty}$ converges to x_1 in $\mathbb{F}$, we can choose $k_1 \in \mathbb{N}$ such that for $k \geq k_1$, $|x_1^{k_1} - x_1| < \frac{1}{2}$.

Once $k_1 \in \mathbb{N}$ has been chosen as above, since $\{x_2^k\}_{k\geq 1}^{\infty}$ converges to x_2 in $\mathbb{F}$, we can select a larger k_2 such that if $k \geq k_2$, $|x_2^{k_2} - x_2| < \frac{1}{4}$. Continuing this way, we can find an increasing sequence of integers $k_1 \leq k_2 \leq \ldots \leq k_n \leq \ldots$ such that

$$\forall j \in \mathbb{N}, \quad |x_n^k - x_n| \leq \frac{1}{2^j} \quad \text{for } n = 1, 2, \ldots, j \text{ and } k \geq k_j \qquad (2.4)$$

In particular

$$|x_n^{k_j} - x_n| \leq \frac{1}{2^j} \quad \text{for all } n \leq j \qquad (2.5)$$

Lemma 1.2.9

The sequence $x = (x_n)_{n=1}^{\infty}$ defined in (2.3) belongs to ℓ^p

P/ Any Cauchy sequence in a normed vector space is bounded (Exercise 1.2.D). In our case, there exist $R < \infty$ such that $\|a^k\|_p \leq R$ for all $k \in \mathbb{N}$. $\qquad (2.6)$

Let $N \in \mathbb{N}$. Write $x_n = (x_n - x_n^{k_N}) + x_n^{k_N}$ and use the triangle inequality of $(\mathbb{F}^N, \| \|_p)$ - Imitate the proof of Prop 1.2.7 - to obtain

$$\left(\sum_{n=1}^{N} |x_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{N} |x_n - x_n^{k_N}|^p\right)^{1/p} + \left(\sum_{n=1}^{N} |x_n^{k_N}|^p\right)^{1/p}$$

$$\leq \left(\sum_{n=1}^{N} |x_n - x_n^{k_N}|^p\right)^{1/p} + \|x^{k_N}\|_p$$

$$\overset{(2.6)}{\leq} \left(\sum_{n=1}^{N} |x_n - x_n^{k_N}|^p\right)^{1/p} + R$$

$$\overset{(2.5)}{\leq} \left(\sum_{n=1}^{N} \left(\frac{1}{2^N}\right)^p\right)^{1/p} + R \qquad (2.7)$$

Since

$$\left(\sum_{n=1}^{N} \frac{1}{2^{Np}}\right)^{1/p} = \left(\frac{N}{2^{Np}}\right)^{1/p} = N^{1/p}/2^N$$

and $\lim_{N\to\infty} N^{1/p}/2^N = 0$, the numerical series $\sum_{n=1}^{\infty} \frac{1}{2^{Np}}$ converges and consequently it is bounded. That is, there exist $S < \infty$ such that $\left(\sum_{n=1}^{N} \left(\frac{1}{2^N}\right)^p\right)^{1/p} \leq S$, $\forall N \in \mathbb{N}$.
Combining this with (2.7) we obtain

$$\left(\sum_{n=1}^{N} |x_n|^p\right)^{1/p} \leq S + R \qquad \text{for all } N \in \mathbb{N}.$$

Since the right-hand side of this inequality does not depend on N, letting $N \to \infty$ we conclude that

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \leq R + S$$

which shows that $x \in \ell_p$. ∎

We need to show II). Let $\varepsilon > 0$. Since $x \in \ell^p$ (lemma 1.2.9) $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Therefore, there is $N_1 \in \mathbb{N}$ such that

$$\sum_{n=N_1}^{\infty} |x_n|^p < \varepsilon^p$$

Also $\{x^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $(\ell^p, \|\ \|_p)$. Hence, there exists $N_2 \in \mathbb{N}$ such that for all $k, l \geq N_2$, $\|x^k - x^l\|_p < \varepsilon$.
If $N = \max\{N_1, N_2\}$

$$\forall\, k \geq N, \quad \sum_{n=N}^{\infty} |x_n|^p < \varepsilon^p \text{ and } \|x^N - x^k\|_p < \varepsilon \qquad (2.8)$$

Since x^N belongs to ℓ^p, we can select N^1 (that can be choosen $N' \geq N$) such that

$$\sum_{n=N'}^{\infty} |x_n^n|^p < \varepsilon^p \qquad (2.9)$$

Since for each $n \in \mathbb{N}$, $\lim_{n\to\infty} x_n^k = x_n$ we can choose $k_1 \in \mathbb{N}$ with $k_1 \geq N'$ s.t.

$$|x_1^k - x_1|^p < \varepsilon^p/_{N'} \quad \text{for } k \geq k_1.$$

Similarly, we can choose $k_2 \in \mathbb{N}$, $k_2 \geq N'$ s.t.

$$|x_2^k - x_2|^p < \varepsilon^p/_{N'} \quad \text{for } k \geq k_2$$

Continuing in this way, after N' steps, we can take $k = \max\{k_1, .., k_{N'}\}$ such that

$$|x_n^k - x_n|^p < \varepsilon^p/_{N'} \quad \text{for } k \geq k \text{ and } n \leq N' \qquad (2.10)$$

Observe that $k \geq N'$ since $k_j \geq N'$ for all $j = 1, 2, .. N'$.
By the triangle inequality for $\|\ \|_p$, if $k \geq k$

$$\|x^k - x\|_p \leq \Big(\sum_{n=1}^{N'-1} |x_n^k - x_n|^p\Big)^{1/p} + \Big(\sum_{n=N'}^{\infty} |x_n^k - x_n|^p\Big)^{1/p} \qquad (2.11)$$

By (2.10)

$$\left(\sum_{n=1}^{N'-1} |x_n^k - x_n|^p\right)^{1/p} \le \left(\sum_{n=1}^{N'-1} \frac{\varepsilon^p}{N'}\right)^{1/p} = \varepsilon\left(\frac{N'-1}{N'}\right)^{1/p} < \varepsilon .$$

For the second term in the right-hand side of (2.11) we use the triangle inequality to obtain

$$\left(\sum_{n=N'}^{\infty} |x_n^k - x_n|^p\right)^{1/p} \le \left(\sum_{n=N'}^{\infty} |x_n^k|^p\right)^{1/p} + \left(\sum_{n=N'}^{\infty} |x_n|^p\right)^{1/p}$$

$$\overset{(2.8)\ (N' \ge N)}{\le} \left(\sum_{n=N'}^{\infty} |x_n^k|^p\right)^{1/p} + \varepsilon$$

Using these last two estimates in (2.11) we obtain

$$\| x^k - x \|_p \le \varepsilon + \left(\sum_{n=N'}^{\infty} |x_n^k|^p\right)^{1/p} + \varepsilon \ , \ k \ge \bar{K} \qquad (2.12)$$

We need to estimate the term in the middle of the right-hand side of (2.11). By the triangle inequality

$$\left(\sum_{n=N'}^{\infty} |x_n^k|^p\right)^{1/p} \le \left(\sum_{n=N'}^{\infty} |x_n^k - x_n^N|^p\right)^{1/p} + \left(\sum_{n=N'}^{\infty} |x_n^N|^p\right)^{1/p}$$

$$\overset{(2.9)}{\le} \left(\sum_{n=1}^{\infty} |x_n^k - x_n^N|^p\right)^{1/p} + \varepsilon$$

$$= \| x^k - x^N \|_p + \varepsilon \overset{(2.8)\ \text{since}\ k \ge \bar{K} \ge N' \ge N}{\le}$$

$$\le \varepsilon + \varepsilon = 2\varepsilon$$

Replacing this estimate in (2.12) we conclude

$$\| x^k - x \|_p \le 4\varepsilon \quad \text{for all } k \ge \bar{K},$$

which shows that $\{x^k\}_{k=1}^{\infty}$ converges to x in $(\ell^p, \| \ \|_p)$.

Q.E.D

Exercise 1.2.D. Let

$$c_0 = \{ x = (x_n)_{n=1}^{\infty} \in \ell^{\infty}(\mathbb{N}) : \lim_{n\to\infty} x_n = 0 \}$$

Prove that $(c_0, \| \|_\infty)$ is a Banach space (Hint: use Prop 1.2.3)

S/ By Prop 1.2.3 it is enough to prove that c_0 is a closed subspace of $(\ell^\infty, \| \|_\infty)$. Let $\{x^n\}_{n=1}^\infty$ a sequence in c_0 with $\lim_{n\to\infty} x^n = x$ with $\| \|_\infty$. We need to prove that $x \in c_0$.

Write $x = (x_k)_{k=1}^\infty$ and $x^n = (x_k^n)_{k=1}^\infty$ $\forall n = 1, 2, \ldots$

Let $\varepsilon > 0$. Since $\{x^n\}_{n=1}^\infty$ converges to x in $\| \|_\infty$, there exist $N_1 = N(\varepsilon) \in \mathbb{N}$ such that if $m \geq N_1$, then $\|x^m - x\|_\infty < \varepsilon/2$. That is

$$|x_k^m - x_k| < \varepsilon/2 \quad \forall\, m \geq N_1, \ \forall\, k \geq 1$$

Also, since $x^{N_1} \in c_0$, there exists $N_2 \in \mathbb{N}$ such that $|x_k^{N_1}| < \varepsilon/2$ $\forall k \geq N_2$. Hence, if $k \geq N_2$,

$$|x_k| \leq |x_k - x_k^{N_1}| + |x_k^{N_1}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This shows that $\lim_{k\to\infty} x_k = 0$ and, hence, $x \in c_0$.

Exercise 1.2.E. Let

$$c_{00} = \{ x = (x_n)_{n=1}^\infty \in \ell^\infty(\mathbb{N}) : \exists N = N(x) \in \mathbb{N} \text{ and } x_k = 0 \ \forall k \geq N \}$$

be the set of sequences of elements of $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ such that the terms of the sequence are all zero at the end of the sequence. $(c_{00}, \| \|_\infty)$ is clearly a normed vector subspace of ℓ^∞. Show that $(c_{00}, \| \|_\infty)$ is not complete.

S/ Let $x^n = (x^n_k)_{k=1}^{\infty}$ given by $x^n_k = \begin{cases} 1/k & \text{if } k \le n \\ 0 & \text{if } k > n \end{cases}$

Clearly each $x^n \in c_{00}$ and $\|x^n\|_\infty = 1$. To see that it is a Cauchy sequence in $\| \ \|_\infty$ observe that if $n < m$,

$$\|x^n - x^m\|_\infty = \sup\left\{\frac{1}{n+1}, \frac{1}{n+2}, \dots, \frac{1}{m}\right\} = \frac{1}{n+1} \xrightarrow{n\to\infty} 0$$

It limit in $(\ell^\infty, \| \ \|_\infty)$ is

$$x = (x_k)_{k=1}^{\infty} \quad \text{with} \quad x_k = 1/k$$

since

$$\|x^n - x\|_\infty = \sup\left\{\frac{1}{n+1}, \frac{1}{n+2}, \dots\right\} = \frac{1}{n+1} \xrightarrow{n\to\infty} 0$$

Just observe that $x \notin c_{00}$

If $x = (x_k)_{k=1}^{\infty}$ is a sequence of elements of $\mathbb{F}$, for each $k = 1, 2, 3, \dots$ $|x_k| \le \sum_{n=1}^{\infty} |x_n| = \|x\|_1$ since $|x_k|$ is one of the terms of the series. Taking supremum over $k = 1, 2, \dots$ we deduce $\|x\|_\infty \le \|x\|_1$ which shows that $\ell^1 \subset \ell^\infty$. Similarly, it can be proved that $\ell^p \subset \ell^\infty$ for all $1 \le p < \infty$.

<u>Exercise</u> 1.2.F Let $1 \le p < q < \infty$.

1) If $\|x\|_p = 1$, show that $\|x\|_q \le 1$

2) Prove that if $x \in \ell^p$, then $\|x\|_q \le \|x\|_p$ (This shows that $\ell^p \subset \ell^q$)

P/ 1) $\|x\|_p = 1 \iff \sum_{k=1}^{\infty} |x_k|^p = 1$ ($x = (x_k)_{k=1}^{\infty}$). In particular, $|x_k|^p \le 1$ $\forall k = 1, 2, 3, \dots$ Since $q > p$,

$$|x_k|^q = (|x_k|^p)^{q/p} \le |x_k|^p$$

Hence

$$\|x\|_q = \left(\sum_{k=1}^{\infty} |x_k|^q\right)^{1/q} \le \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/q} = 1^{1/q} = 1.$$

2) If $x=0$ there result is obvious. Assume $x \neq 0$, $x \in \ell^p$.
Apply 1) to the sequence $y = \frac{x}{\|x\|_p} = \left(\frac{x_k}{\|x\|_p}\right)_{k=1}^{\infty}$. Observe that $\|y\|_p = \frac{1}{\|x\|_p}\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p} = 1$. Therefore $\|y\|_q \leq 1$.
This is equivalent to

$$\left\| \frac{x}{\|x\|_p} \right\|_q \leq 1 \iff \|x\|_q \leq \|x\|_p .$$