1. Bases in Banach Spaces

1.1. Banach spaces: definitions and first results

We want to extend the notion of a bases of a finite-dimensional vector space. In a vector space of dimension n, a basis β is a collection of n linearly independent vectors: $\beta = \{e_1, .., e_n\}$ are l.ind. if $\sum_{i=1}^{n} \lambda_i e_i = 0$ $(\lambda_i \in \mathbb{C}) \Rightarrow \lambda_i = 0 \ \forall i = 1, .. n$

In a Hilbert space $(\mathbb{H}, \langle , \rangle)$, a collection $\{e_n\}_{n\geq 1}^{\infty}$ is an orthonormal system if $\langle e_n, e_m \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$. This collection is said to be a basis, if $\overline{\text{span}\{e_n\}_{n=1}^{\infty}} = \mathbb{H}$.

We recall that if $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathbb{H}$, then for every $x \in \mathbb{H}$

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$$

and this representation of x as an infinite sum is unique

This will be our model for the definition of a basis in a Banach space.

A vector space $\mathbb{B}$ over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is a set with an internal operation $+ : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ such that

1) $\forall x, y \in \mathbb{B}$, $x + y = y + x$ (Commutative)
2) $\forall x, y, z \in \mathbb{B}$, $(x+y)+z = x+(y+z)$ (Associative)
3) $\exists\ 0 \in \mathbb{B}$ s.t. $x + 0 = x$ $\forall x \in \mathbb{B}$
4) For every $x \in \mathbb{B}$ there exists $-x \in \mathbb{B}$ s.t. $x + (-x) = 0$

and an external operation $\cdot : \mathbb{F} \times \mathbb{B} \to \mathbb{B}$ s.t.

5) $\forall \lambda \in \mathbb{F}$, $\forall x, y \in \mathbb{B}$, $\lambda \cdot (x+y) = \lambda x + \lambda y$
6) $\forall x \in \mathbb{B}$, $1 \cdot x = x$

Def 1.1.1. A norm in a vector space $\mathbb{B}$ is a map

$$\| \ \| : \mathbb{B} \to \mathbb{R}^+ \cup \{0\}$$

such that

1) $\|x\| = 0 \iff x = 0$
2) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{F}$, $x \in \mathbb{B}$
3) (Triangle inequality) : $\forall x, y \in \mathbb{B}$, $\|x+y\| \leq \|x\| + \|y\|$.

In a normed vector space $(\mathbb{B}, \| \ \|)$ the map $d : \mathbb{B} \times \mathbb{B} \to \mathbb{R}^+ \cup \{0\}$ given by $d(x,y) = \|x - y\|$ has the following properties

1) $d(x,y) = 0 \iff x = y \quad \forall x, y \in \mathbb{B}$

   (Follows from 1) of Def 1.1.1)

2) $d(x,y) = d(y,x) \quad \forall x, y \in \mathbb{B}$

   (Follows from 2) of Def 1.1.1)

3) $d(x,y) \leq d(x,z) + d(z,x) \quad \forall x, y, z \in \mathbb{B}$ (triangle inequality)

   P/ $d(x,y) = \|x - y\| = \|(x-z) + (z+y)\| \leq \|x - z\| + \|z - y\|$

   $= d(x,z) + d(z,y)$ by 3) of Def 1.1.1.

In a normed vector space $(\mathbb{B}, \| \ \|)$ the (open) ball of radius $r > 0$ centered at $x \in \mathbb{B}$ is

$$B_r(x) = \{y \in \mathbb{B} : d(x,y) < r\} = \{y \in \mathbb{B} : \|x - y\| < r\}.$$

The collection $\{B_r(x) : x \in \mathbb{B}, r > 0\}$ is a basis for a topology in $\mathbb{B}$. The open sets are the unions of (open) balls.

Exercise 1.1.1. A. $(\mathbb{B}, \| \ \|)$ normed vector space

a) Show that $\forall x, y \in \mathbb{B}$, $\big| \|x\| - \|y\| \big| \leq \|x-y\|$

b) Show that $\| \ \| : \mathbb{B} \to \mathbb{R}$ is continuous

S/ a) To prove $-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|$

- $\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\| \Rightarrow \|x\| - \|y\| \leq \|x-y\|$
- $\|y\| = \|y-x+x\| \leq \|y-x\| + \|x\| \Rightarrow -\|x-y\| \leq \|x\| - \|y\|$

b) Observe that $\mathbb{R}$ is a normed vector space with norm given by absolute value. Therefore, it is a topological vector space and we can talk about continuity.

Given $\varepsilon > 0$ y $\|x-y\| < \varepsilon$, then by a) $\big| \|x\| - \|y\| \big| \leq \varepsilon$.

Def 1.1.2. $(\mathbb{B}, \| \ \|)$ normed vector space. A sequence $\{x_n\}_{n=1}^{\infty}$ is called a <u>Cauchy</u> sequence if $\forall \varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ s.t. for all $n, m \geq N$, $\|x_n - x_m\| < \varepsilon$

We say that a sequence $\{x_n\}_{n=1}^{\infty}$ in a normed vector space converges to $x \in \mathbb{B}$, and write $\lim_{n\to\infty} x_n = x$ if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ s.t. for all $n \geq N$,

$$\|x_n - x\| < \varepsilon.$$

Def 1.1.3

a) A norm $\| \ \|$ is a vector space $\mathbb{B}$ is called <u>complete</u> if every Cauchy sequence converges to an element of $\mathbb{B}$

b) A vector space $\mathbb{B}$ with a complete norme is called a <u>Banach space</u>

Let S be a set contained in a normed space $\mathbb{B}$, with norm $\| \ \|$. S is said to be <u>dense</u> in $\mathbb{B}$ if for every non-empty open set W in $\mathbb{B}$, $S \cap W \neq \emptyset$

Theorem 1.1.4 (Baire's theorem)

> Let $(\mathbb{B}, \| \ \|)$ be a Banach space. The intersection of every countable collection of dense open subsets of $\mathbb{B}$ is dense in $\mathbb{B}$.

P/ Let $\{V_n\}_{n=1}^{\infty}$ be a countable collection of dense open sets in $\mathbb{B}$. Let $S = \bigcap_{n=1}^{\infty} V_n$. Let W be any nonempty open set in $\mathbb{B}$. We have to show that $S \cap W \neq \emptyset$

Since V_1 is dense in $\mathbb{B}$, by definition $W \cap V_1 \neq \emptyset$. Since V_1 and W are open, $V_1 \cap W$ is open. We can find $x_1 \in W \cap V_1$ and $0 < r_1 < 1$ such that $\overline{B_{r_1}(x_1)} \subset V_1 \cap W$ (1.1)

Suppose that for $n \geq 2$ we have found $x_2, \dots, x_{n-1}$ and $r_2, \dots, r_{n-1}$ with $0 < r_j < \frac{1}{j}$ & $\overline{B_{r_j}(x_j)} \subset V_j \cap B_{r_{j-1}}(x_{j-1})$, $j = 2, \dots, n-1$

Since V_n is dense in $\mathbb{B}$, $V_n \cap B_{r_{n-1}}(x_{n-1}) \neq \emptyset$ and open. We can find $x_n \in V_n \cap B_{r_{n-1}}(x_{n-1})$ and $0 < r_n < \frac{1}{n}$ such that

$$\overline{B_{r_n}(x_n)} \subset V_n \cap B_{r_{n-1}}(x_{n-1}) \qquad (1.2)$$

Observe that the balls $B_{r_n}(x_n)$ are nested: $B_{r_n}(x_n) \subset B_{r_{n-1}}(x_{n-1})$ $n = 1, 2, \dots$. Therefore, we have found a sequence $\{x_n\}_{n=1}^{\infty}$ such that if $i, j > n$,

$$x_i \in B_{r_{i-1}}(x_{i-1}) \subset B_{r_n}(x_n) \quad \text{and} \quad x_j \in B_{r_{j-1}}(x_{j-1}) \subset B_{r_n}(x_n) \qquad (1.3)$$

Hence, $d(x_i, x_j) = \|x_i - x_j\| \le \|x_i - x_n\| + \|x_n - x_j\| \le r_n + r_n$ $= 2r_n < 2/n$. This shows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathbb{B}, \|\ \|)$. Since $(\mathbb{B}, \|\ \|)$ is complete, there exists $x \in \mathbb{B}$ such that $\lim_{n\to\infty} x_n = x$.

Let $n \in \mathbb{N}$. For $i > n$, $x_i \in \overline{B_{r_n}(x_n)}$ by (1.3). Since $\lim_{n\to\infty} x_n = x$, and $\overline{B_{r_n}(x_n)}$ is closed, $x \in \overline{B_{r_n}(x_n)}$. By (1.2), $x \in V_n$.

With $n=1$, $x \in \overline{B_{r_1}(x_1)}$. By (1.1), $x \in W$.

Hence $x \in (\bigcap_{n=1}^{\infty} V_n) \cap W = S \cap W$ and $S \cap W \neq \emptyset$. QED

For a sequence $\{x_n\}_{n=1}^{\infty}$ in a normed vector space $(\mathbb{B}, \|\ \|)$ the <u>partial sums</u> of the formal series $\sum_{n=1}^{\infty} x_n$ are defined as

$$S_n := \sum_{k=1}^{n} x_k$$

Def 1.1.5. $\{x_n\}_{n=1}^{\infty}$ sequence in a normed vector space $(\mathbb{B}, \|\ \|)$. We say that the series $\sum_{n=1}^{\infty} x_n$ converges to $x \in \mathbb{B}$ if $\lim_{n\to\infty} S_n = x$. If this is the case, we write

$$\sum_{n=1}^{\infty} x_n = x$$

Def 1.1.6 A series $\sum_{n=1}^{\infty} x_n$ of elements of of a normed vector space $(\mathbb{B}, \|\ \|)$ is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\| < \infty$ (that is, the series of real numbers is convergent in $\mathbb{R}$)

Theorem 1.1.7. (Completeness criterion with series).

Let $(\mathbb{B}, \|\ \|)$ be a normed vector space. The following are equivalent

(a) $(\mathbb{B}, \|\ \|)$ is a Banach space

(b) Every absolutely convergent series in $\mathbb{B}$ converges to an element of $\mathbb{B}$.

P/ (a) $\Rightarrow$ (b) Suppose that $(\mathbb{B}, \|\ \|)$ is a Banach space and consider a series $\sum_{n=1}^{\infty} x_n$ in $\mathbb{B}$ such that

$$\sum_{n=1}^{\infty} \|x_n\| < \infty . \tag{1.4}$$

Let $S_n = \sum_{k=1}^{n} x_n$ be the partial sums of the series $\sum_{k=1}^{\infty} x_k$.

We want to show that $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{B}$. For $\varepsilon > 0$, since every convergent sequence in $\mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$, there exists $N = N(\varepsilon) \in \mathbb{N}$ s.t.

$$\sum_{k=n+1}^{m} \|x_k\| < \varepsilon \quad \forall\ m > n \geq N \tag{1.5}$$

Therefore, if $m > n > N$, by the triangle inequality

$$\|S_m - S_n\| = \left\| \sum_{k=n+1}^{m} x_k \right\| \leq \sum_{k=n+1}^{m} \|x_k\| < \varepsilon$$

This shows that $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathbb{B}, \|\ \|)$.

Since $(\mathbb{B}, \|\ \|)$ is complete, there exists $x \in \mathbb{B}$ such that

$\lim_{n\to\infty} S_n = x$. By Definition 1.1.5, $\sum_{n=1}^{\infty} x_n = x$ in $\mathbb{B}$.

(b) $\Rightarrow$ (a) Suppose that $(\mathbb{B}, \|\ \|)$ is not complete. We will construct a divergen serie $\sum_{n=1}^{\infty} w_n$ in $\mathbb{B}$ wich is absolutely convergent.

Since $(\mathbb{B}, \|\ \|)$ is not complete, there exist a sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathbb{B}$ which is a Cauchy sequence, but does not converge to an element of $\mathbb{B}$.

Exercise 1.1.B If $\{V_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a normed vector space $(\mathbb{B}, \|\ \|)$, there exists a subsequence $\{V_{n_k}\}_{k=1}^{\infty}$ such that $\|V_{n_{k+1}} - V_{n_k}\| < \frac{1}{2^k}$, $k=1,2,3,\ldots$

P/ Since $\{V_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathbb{B}, \|\ \|)$, for $\varepsilon = 1$, there exists $n_1 \in \mathbb{N}$ s.t.

$$\|V_{n_1} - V_n\| \le 1 \quad \forall\, n \ge n_1 \qquad (1.B.1)$$

For $\varepsilon = 1/2$, choose $n_2 \in \mathbb{N}$ s.t. $n_2 > n_1$ and

$$\|V_{n_2} - V_n\| \le 1/2 \quad \forall\, n \ge n_2 \qquad (1.B.2)$$

By (1.B.1), $\|V_{n_1} - V_{n_2}\| \le 1$ since $n_2 > n_1$.

Suppose we have choosen $V_{n_1}, V_{n_2}, \ldots, V_{n_k}$ such that $n_1 < n_2 < \ldots < n_k$, $\|V_{n_\ell} - V_{n_{\ell+1}}\| < 1/2^\ell$, $\ell = 1, 2, \ldots, k-1$, and

$$\|V_{n_k} - V_n\| \le \frac{1}{2^k} \quad \forall\, n \ge n_k \qquad (1.B.3)$$

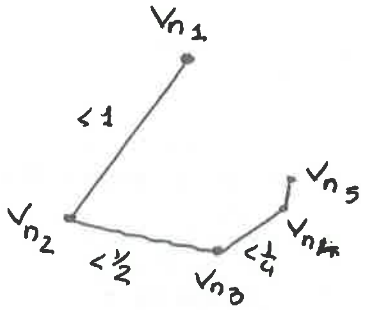
For $\varepsilon = \frac{1}{2^{k+1}}$, choose $n_{k+1} \in \mathbb{N}$ s.t. $n_{k+1} > n_k$ and

$$\|V_{n_k} - V_n\| \le \frac{1}{2^{k+1}} \quad \forall\, n \ge n_{k+1}.$$

By (1.B.3), since $n_{k+1} > n_k$,

$$\|V_{n_k} - V_{n_{k+1}}\| \le \frac{1}{2^k}.$$

QED



Define $x_k = V_{n_{k+1}} - V_{n_k}$, $k = 1, 2, 3, \ldots$ By Exercise 1.1.B,

$$\sum_{k=1}^{\infty} \|x_k\| = \sum_{k=1}^{\infty} \|V_{n_{k+1}} - V_{n_k}\|$$

$$\le \sum_{k=1}^{\infty} 1/2^k = 2 < \infty$$

Hence, the series $\sum_{k=1}^{\infty} x_{n_k}$ is absolutely convergent in $(\mathbb{B}, \|\ \|)$.
On the other hand, $\sum_{k=1}^{\infty} x_{n_k}$ does not converge in $(\mathbb{B}, \|\ \|)$ because its partial sums are

$$S_m = \sum_{k=1}^{m} x_{n_k} = x_{n_1} + \ldots + x_{n_m} =$$

$$= (V_{n_2} - V_{n_1}) + (V_{n_3} - V_{n_2}) + \ldots + (V_{n_m} - V_{n_{m-1}})$$

$$= V_{n_m} - V_{n_1}$$

and $\{V_n\}_{n=1}^{\infty}$ does not converge in $(\mathbb{B}, \|\ \|)$. See Exercise 1.1.C.

QED

<u>Exercise</u> 1.1.C. Let $\{V_n\}_{n=1}^{\infty}$ be a Cauchy sequence in a normed vector space $(\mathbb{B}, \|\ \|)$. If there exists a subsequence $\{V_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} V_{n_k} = x \in \mathbb{B}$, then $\lim_{n\to\infty} V_n = x$.