Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2021-22 Homework 5
Due: Friday, May 20, 2022

Frame theory.

1. Let $\{\varphi_k : k = 1, 2, \dots\}$ be a Parseval frame in a Hilbert space $\mathbb{H}$. Show that the following conditions are equivalent:
a) $\{\varphi_k : k = 1, 2, \dots\}$ is an orthonormal basis of $\mathbb{H}$.
b) $\|\varphi_k\| = 1$ for all $k = 1, 2, \dots$.

2. a) Let $\varphi_k = (a_k \cos\theta_k, a_k \sin\theta_k)\,, k = 1, 2, \dots, M$ $(M \geq 2)$ be vectors in $\mathbb{R}^2$ written in polar coordinates. Prove that $\Phi = \{\varphi_k : k = 1, 2, \dots, M\}$ is a tight frame for $\mathbb{R}^2$ if and only if

$$\sum_{k=1}^{M} a_k^2 \cos 2\theta_k = 0, \quad \text{and} \quad \sum_{k=1}^{M} a_k^2 \sin 2\theta_k = 0\,.$$

(Hint: Write the synthesis operator T in matrix form and use that Φ is a tight frame with constant A if and only if $F = TT^* = AI$, where F is the frame operator.)
b) Show that if $n \geq 2$ the n^{th}-roots of unity, that is the vertices of and n-sided regular polygon, form a tight frame for $\mathbb{R}^2$.

3. Let $\Phi = \{\varphi_k : k = 1, 2, \dots,\}$ be a frame in a separable Hilbert space $\mathbb{H}$, with frame operator F. Since F is a positive, selfadjoint and invertible operator, so is F^{-1}. Its positive square root, denoted by $F^{-1/2}$, is also positive and selfadjoint, and commutes with F. Show that $\Psi = \{\psi_k = F^{-1/2}\varphi_k : k = 1, 2, \dots,\}$ is a Parseval frame.

4. a) For $g \in L^2(\mathbb{R})$, let $\mathcal{G}(g) = \{M_m T_k g : m, k \in \mathbb{Z}\}$ be a frame for $L^2(\mathbb{R})$. Prove that the frame operator F of the frame $\mathcal{G}(g)$ as well as its inverse commute with modulations M_n and translations T_l.
b) Let $\psi \in L^2(\mathbb{R})$. Suppose that $W(\psi) = \{D_{2^j} T_k \psi : j, k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. Show that its frame operator F as well as its inverse commute with dilations $D_{2^l} f(x) = 2^{\ell/2} f(2^\ell x)\,, \ell \in \mathbb{Z}$.

5. Suppose that for $g \in L^2(\mathbb{R})$, the collection $\mathcal{G}(g) = \{M_m T_k g : m, k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B. Show that

$$A \leq |\mathcal{Z}g(x,\xi)|^2 \leq B\,, \qquad a.e\ (x,\xi) \in [0,1)^2\,,$$

where $\mathcal{Z}g$ denotes the Zak transform of g.
(Hint: Start proving $\mathcal{Z}(M_m T_k g)(x,\xi) = e^{2\pi i m x} e^{-2\pi i k \xi} \mathcal{Z}g(x,\xi)$. Then show the equality

$$\sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f, M_m T_k g\rangle|^2 = \int_0^1 \int_0^1 |\mathcal{Z}g(x,\xi)|^2 |\mathcal{Z}f(x,\xi)|^2 dx d\xi\,.$$

Use a measure theoretic argument and the definition of frame to show the result.)