Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2021-22 Homework 4
Due: Wednesday, May 4, 2022

Explicit constructions of wavelets.

1. A scaling function of the Haar MRA is $\varphi = \chi_{[0,1]}$ and $\mathcal{F}\varphi(\xi) = e^{-\pi i\xi}\dfrac{\sin(\pi\xi)}{\pi\xi}$. The low pass filter of the Haar MRA is $h(\xi) = e^{-\pi i\xi}\cos(\pi\xi)$. Therefore, the following formula must hold:

$$\prod_{j=1}^{\infty} e^{-\pi i\xi/2^j}\cos(\frac{\pi\xi}{2^j}) = e^{-\pi i\xi}\frac{\sin(\pi\xi)}{\pi\xi}.$$

Show this formula directly using the trigonometric relation $\sin(2\alpha) = 2(\sin\alpha)(\cos\alpha)$.

2. Let $h(\xi) = e^{-3\pi i\xi}\cos(3\pi\xi)$. Clearly, h is a continuous function on $\mathbb{R}$, it is one periodic, and $h(0) = 1$.

a) Show that

$$|h(\xi)|^2 + h(\xi + 1/2)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}.$$

b) Define φ by $\mathcal{F}\varphi(\xi) = \prod_{j=1}^{\infty} h(\xi/2^j)$. Show that $\varphi = \dfrac{1}{3}\chi_{[0,3]}$. (Observe that the integer translates of φ do not form an orthogonal system in $L^2(\mathbb{R})$.)

3. Let c_k be given by

$$\frac{1}{c_k} = \int_0^{1/2} (\sin 2\pi x)^{2k+1} dx > 0,$$

and

$$g_k(\xi) = 1 - c_k \int_0^{\xi} (\sin 2\pi x)^{2k+1} dx\,.$$

a) Show that

$$g_0(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^2.$$

b) Show that

$$g_1(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^4 (2 - \cos 2\pi\xi)\,.$$

(Hint: Write $\sin^2 x = (1 + \cos x)(1 - \cos x)$ and integrate by parts.)

4. (2 puntos) Find the coefficients of a low pass filter of Daubechies ${}_2\psi$ orthonormal wavelet using the polynomial

$$g_1(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^4 (2 - \cos 2\pi\xi)\,,$$

found in the previous exercise.

5. (2 puntos) Let c_k be given by

$$\frac{1}{c_k} = \int_0^{1/2} (\sin 2\pi x)^{2k+1} dx > 0,$$

and

$$g_k(\xi) = 1 - c_k \int_0^{\xi} (\sin 2\pi x)^{2k+1} dx\,.$$

Show, integrating by parts, that

$$g_k(\xi) = \left|\frac{1+e^{-2\pi i\xi}}{2}\right|^{2k+2} P_k(\xi)\,,$$

where

$$P_k(\xi) = \frac{1}{2^k}\sum_{\ell=0}^{k} \binom{2k+1}{k-\ell}(1+\cos 2\pi\xi)^{\ell}(1-\cos 2\pi\xi)^{k-\ell}$$

is and even trigonometric polynomial with real coefficients.

Coding and Entropy.

6. Show that for any source of information X with N symbols

$$0 \le \mathcal{E}(X) \le \log_2 N,$$

where $\mathcal{E}(X) = \sum_{k=1}^{n} p_k \log_2 \frac{1}{p_k}$ is the Shannon entropy of X for the probabilities $p_k, k = 0, 1, 2, \ldots, N$. (Hint: Use Lagrange multipliers to show that the maxima of $\mathcal{E}(X)$ in the region $0 \le p_k \le 1, k = 1, 2, \ldots, N$, with the restriction $\sum_{k=1}^{N} p_k = 1$ is attained at $p_k = 1/N, k = 1, 2, \ldots, N$.

7. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ be a source of information whose probabilities are

$$p_1 = 0.49,\;\; p_2 = 0.26,\;\; p_3 = 0.12, p_4 = 0.04,\;\; p_5 = 0.04,\;\; p_6 = 0.03, p_7 = 0.02.$$

a) Compute the entropy of X. Build a binary Huffman code $\mathcal{C}$ for these probabilities and compute $\mathcal{A}_{\mathcal{C}}(X)$.

b) Suppose that the symbols of X are codify with a ternary code that takes values 0, 1, and 2. The code can now be represented in a ternary tree. Extend Huffman algorithm to find a ternary tree for X having the prefix condition and compute the average length of its codified words.

8. The quantize coefficients of an 8×8 block of an image are

$$\begin{bmatrix} -26 & -3 & -6 & 2 & 2 & -1 & 0 & 0 \\ 0 & -2 & -4 & 1 & 1 & 0 & 0 & 0 \\ -3 & 1 & 5 & -1 & -1 & 0 & 0 & 0 \\ -4 & 1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a binary Huffman code to represent these symbols.