Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2021-22

Homework 3
Due: Friday, April 29, 2022

Wavelets and Multiresolution Analysis.

1. Find the Haar coefficients, that is, $< f, \psi_{j,k} >$ for all $j, k \in \mathbb{Z}$, for the function $f = \chi_{[0,1)}$, where ψ is the Haar wavelet.

2. Show that for the Haar-MRA with scaling function $\varphi = \chi_{[0,1)}$, its low pass filter coefficients are given by $h(0) = h(1) = 1/2$ and $h(k) = 0$ if $k \neq 0, 1$, and the low pas filter is $h(\xi) = e^{-\pi i \xi} \cos(\pi \xi)$.

3. Show that for the Shannon-MRA with scaling function φ given by $\varphi(x) = \dfrac{\sin(\pi x)}{\pi x}, x \neq 0$ and $\varphi(0) = 1$ (recall that $\mathcal{F}\varphi = \chi_{[-1/2,1/2)}$) the associate discrete filter is

$$h(k) = \begin{cases} 0 & \text{if } k = 2\ell, \ell \neq 0 \\ 1/2 & \text{if } k = 0 \\ \frac{(-1)^\ell}{\pi(2\ell+1)} & \text{if } k = 2\ell + 1 \end{cases}$$

Show that the low pass filter of this MRA is

$$h(\xi) = \sum_{k=-\infty}^{\infty} \chi_{[-\frac{1}{4},\frac{1}{4})}(\xi + k)\,.$$

4. Let $h(\xi)$ and $g(\xi)$ be two 1-periodic functions in $L^2([0,1))$ satisfying $g(\xi) = e^{-2\pi i \xi}\overline{h(\xi + \frac{1}{2})}$. If $h(\xi) = \sum_{k=-\infty}^{\infty} h(k) e^{-2\pi i k \xi}$ and $g(\xi) = \sum_{k=-\infty}^{\infty} g(k) e^{-2\pi i k \xi}$, show that

$$g(k) = \overline{h(1-k)}(-1)^{1-k}\,, \quad k \in \mathbb{Z}\,.$$

5. Consider the Haar-MRA with scaling function $\varphi = \chi_{[0,1)}$. Show that an orthonormal wavelet associated to this MRA is given by $\psi(x) = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}$.

6. For the Shannon MRA with scaling function $\mathcal{F}\varphi = \chi_{[-\frac{1}{2},\frac{1}{2})}$, show that one of this associated orthonormal wavelets is given by

$$\mathcal{F}\psi(\xi) = e^{-\pi i \xi} \chi_{[-1,-\frac{1}{2}) \cup [\frac{1}{2},1)}\,.$$

7. a) For a positive integer $p, p > 1$, write the definition of MRA(p), that is a Multiresolution Analysis with dilation factor p.

b) Show that for an MRA(p), there exists a 1-periodic function $h(\xi)$ belonging to $L^2([0,1))$ such that

$$\mathcal{F}\varphi(\xi) = h(\frac{\xi}{p})\mathcal{F}(\frac{\xi}{p}),$$

where φ is the scaling function of the MRA(p). The function h is called the low pass filter of the MRA(p).

8. For and MRA(p), $p > 1$, show that

$$\sum_{s=0}^{p-1} |h(\xi + \frac{s}{p})|^2 = 1 \quad \text{a.e } \xi \in \mathbb{R},$$

where h is the low pass filter of the MRA(p) whose existence is proved in part b) of the previous exercise.

9. Show that the detail coefficients $d_{j-1}(p)$ are computed with the formula

$$d_{j-1}(p) = \sqrt{2}\sum_{k\in\mathbb{Z}} \overline{g(k-2p)}c_j(k),$$

where $\{c_j(k) : k \in \mathbb{Z}\}$ are the coefficients of f at level j and $\{g(k) : k \in \mathbb{Z}\}$ are the high pass filter coefficients.

10. For the Haar wavelet, show that the decomposition algorithm is

$$c_{j-1}(p) = \sqrt{2}\,\frac{c_j(2p) + c_j(2p+1)}{2}, \quad d_{j-1}(p) = \sqrt{2}\,\frac{-c_j(2p) + c_j(2p+1)}{2},$$

and the reconstruction algorithm is

$$c_j(2p) = \sqrt{2}\,\frac{c_{j-1}(p) - d_{j-1}(p)}{2}, \quad c_j(2p+1) = \sqrt{2}\,\frac{c_{j-1}(p) + d_{j-1}(p)}{2}.$$