Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2021–22 Homework 2
Due: Wednesday, March 23, 2022

Orthonormal bases for signal and image processing.

1. Given $f : [0,1] \longrightarrow \mathbb{R}$ with $f \in L^2([0,1])$, extend f to $\mathbb{R}$ to obtain a function $\widetilde{f}$ odd with respect to the origin, even with respect to 1 and -1 and 4-periodic. By writing the Fourier series of $\widetilde{f}$ in $[-2,2]$ in terms of sines and cosines show that the cosine coefficients are zero as well as the even sine coefficients. Prove that

$$\{\sqrt{2}\sin(\frac{2k+1}{2}\pi x) : k = 0,1,2,\dots\}$$

is and orthornormal basis of $L^2([0,1])$. This is called the **sine-IV** basis for $L^2([0,1])$.

2. Show that for a function $f \in L^2(\mathbb{R}) \cap C^2(\mathbb{R})$, the coefficients of f in the block cosine-I basis given by

$$\{\chi_{[n,n+1)}(x) : n \in \mathbb{Z}\} \cup \{\chi_{[n,n+1)}(x)\sqrt{2}\cos \pi k(x-n) : n \in \mathbb{Z}, k = 1,2,\dots\}$$

decay, for n fixed, at a rate proportional at least to $1/k^2$.

3. Given $\varepsilon > 0$, choose ψ an even, C^∞ function defined on $\mathbb{R}$, supported on $[-\varepsilon,\varepsilon]$ such that $\int_\varepsilon^\varepsilon \psi(x) = \pi/2$. Let $\theta(x) = \int_{-\infty}^x \psi(y)dy$. Show that $\theta(x)+\theta(-x) = \pi/2$. Define $s_\varepsilon(x) = \sin\theta(x)$. Show that $[s_\varepsilon(x)]^2 + [s_\varepsilon(-x)]^2 = 1$.

4. With the same notation as in the previous exercise, let $c_\varepsilon(x) = \cos(\theta(x))$. Let $I = [\alpha,\beta] \subset \mathbb{R}, \varepsilon, \varepsilon' > 0$, such that $\alpha + \varepsilon < \beta - \varepsilon'$. The function

$$b_I(x) = s_\varepsilon(x-\alpha)c_{\varepsilon'}(x-\beta)$$

is called a **bell** function associated with the interval $I = [\alpha,\beta]$.

a) Sketch the graph of the bell function b_I.

b) Show that on $[\alpha-\varepsilon, \alpha+\varepsilon]$

- i) $b_I(x) = s_\varepsilon(x-\alpha)$.
- ii) $b_I(2\alpha - x) = s_\varepsilon(\alpha - x) = c_\varepsilon(x-\alpha)$.
- iii) $b_I^2(x) + b_I^2(2\alpha - x) = 1$.

5. Show that the collection of N vectors

$$\mu_k \frac{1}{\sqrt{N}}\Big(\sin\frac{k\pi}{N}(n+\frac{1}{2})\Big)_{n=-N}^{N-1}, \qquad k = 1,2,\dots,N,$$

each one of size $2N$, where $\mu_k = 1$ if $k = 1,2,\dots,N-1$ and $\mu_N = 1/\sqrt{2}$, is an orthonormal system.

6. Show that the N vectors given by

$$\mu_k\sqrt{\frac{2}{N-1}}\Big(\lambda_n \cos[\frac{\pi}{N-1}\,k\,n]\Big)_{n=0}^{N-1}, \qquad k=0,1,2,\dots,N-1\,,$$

each one of size N, where $\lambda_0 = 1/\sqrt{2}$, $\lambda_{N-1} = 1/\sqrt{2}$ and $\lambda_n = 1$ if $n = 1,2,\dots,N-2$, and $\mu_0 = \mu_{N-1} = 1/\sqrt{2}$, and $\mu_k = 1$ if $k = 1,2,\dots,N-2$, is an orthonormal basis of the space of signals of size N. This basis corresponds to the one obtained by extending $f = (f(n))_{n=0}^{N-1}$ evenly with respect to $n = 0$.

7. (2 points) This exercise shows how to calculate DCT-I with an induction relation that involves DCT-IV.

a) Regroup the terms $f(n)$ and $f(N-1-n)$, $0 \le n \le \dfrac{N}{2}-1, N = 2^q$, in the DCT-I, to write $\widehat{f}_I(2k)$ as the DCT-I of the signal

$$s(n) = \frac{1}{\sqrt{2}}[f(n) + f(N-1-n)], \qquad 0 \le n \le \frac{N}{2}-1.$$

b) With the same technique as in part a), write $\widehat{f}_I(2k+1)$ as the DCT-IV of the signal

$$r(n) = \frac{1}{\sqrt{2}}[f(n) - f(N-1-n)], \qquad 0 \le n \le \frac{N}{2}-1.$$

c) Using that, with a fast algorithm, the number of operations to calculate DCT-IV of size N is $O(N\log_2 N)$ and parts a) and b), show that with the above algorithm, the number of operations needed to calculate DCT-I of size N is also $O(N\log_2 N)$.

8. Show that the number $B_j^{(2)}$ of orthogonal bases of the space of discrete images of size $N^2 (N = 2^L)$ in a bi-dyadic tree of depth $j, 1 \le j \le L$, satisfies

$$2^{4^{j-1}} \le B_j^{(2)} \le 2^{\frac{4}{3}4^{j-1}}.$$

9. Consider the signal f of size $N = 8$ given by

$$f = (8, 16, 24, 32, 40, 48, 56, 64)\,.$$

a) Compute the DCT-I of f, rounding the result to the nearest integer. Compress the signal 50% by setting to zero the DCT-I coefficients in positions 4, 5, 6, and 7. Find now the inverse DCT-I of this compressed signal, and, after rounding, observe that is similar to the original one.

b) Take now the orthonormal basis of $\mathbb{C}^8$ given by

$$\Big\{\lambda_k\frac{1}{2}\Big(\cos\frac{\pi k n}{4}\Big)_{n=0}^{7}\Big\}_{k=0}^{4}\bigcup\Big\{\frac{1}{2}\Big(\sin\frac{\pi k n}{4}\Big)_{n=0}^{7}\Big\}_{k=1}^{3}.$$

where $\lambda_0 = \lambda_4 = \dfrac{1}{\sqrt{2}}$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Repeat the process in a), setting now to zero the frequencies $k = 3$ and $k = 4$ of cosines, and the frequencies $k = 2$ and $k = 3$ of sines. Observe that the final result is somehow different than the original signal.