Agostinho Neto University College of Sciences

Master in Mathematics and Applications

Test 1: Wavelets and signal processing

14 / 07 / 2021

NAME: ______________________________

N. 1	N. 2	N. 3	N. 4	TOTAL

1. (5 points)

(a) Give the definition of inner product $\langle\ ,\ \rangle$ in a vector space $\mathbb{H}$.

Let $\alpha = (\alpha_n)_{n=1}^{\infty}$ be a sequence of positive real numbers. Consider

$$\ell^2_\alpha(\mathbb{N}) = \{z = (z_n)_{n=1}^{\infty} : z_n \in \mathbb{C} \text{ and } \|z\|_{\ell^2_\alpha} = \left(\sum_{n=1}^{\infty} |z_n|^2 \alpha_n\right)^{1/2} < \infty\}.$$

(b) Prove that $\langle z, w\rangle = \sum_{n=1}^{\infty} z_n \overline{w_n} \alpha_n$ defines an inner product in $\ell^2_\alpha(\mathbb{N})$.

(c) Find and orthonormal basis of $\ell^2_\alpha(\mathbb{N})$. (You must show that your example is an orthonormal basis)

2. (5 points)

(a) Give the definition of orthonormal system in a Hilbert space $(\mathbb{H}, \langle\ ,\ \rangle)$.

(b) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in a Hilbert space $(\mathbb{H}, \langle\ ,\ \rangle)$. Prove Bessel's inequality, that is, for all $x \in \mathbb{H}$,

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2 .$$

Continues in reverse side

3. (5 `points`) Let $(\mathbb{H}, \langle\ ,\ \rangle)$ be a Hilbert space and $\{e_n\}_{n=1}^{\infty}$ a sequence of elements of $\mathbb{H}$ of norm 1, that is, $\|e_n\| = 1$ for all $n \in \mathbb{N}$. Suppose that for all $x \in \mathbb{H}$,

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = \|x\|^2 .$$

(a) Show that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal system of $\mathbb{H}$.

(b) For $x \in \mathbb{H}$ and $N \in \mathbb{N}$ define $x_N = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$. Prove that $\{x_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $\mathbb{H}$. (Hint: use that for $x \in \mathbb{H}$, $\|x\| = \sup_{\|y\|=1} |\langle x, y\rangle|$)

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4. (5 `points`)

(a) Give the definitions of Fourier coefficients $\widehat{f}(k)$, $k \in \mathbb{Z}$, and Fourier series of a 1-periodic function $f \in L^1_p([-1/2, 1/2])$.

Let

$$f(x) = x^2 , \quad -\frac{1}{2} \leq x < \frac{1}{2},$$

extended to $\mathbb{R}$ with period one.

(b) Compute its Fourier coefficients $\widehat{f}(k)$ for all $k \in \mathbb{Z}$, starting with $\widehat{f}(0)$.