Agostinho Neto University – College of Sciences
Master in Mathematics and Applications
Wavelets and Signal Processing - 2021 Review Exercises

Hilbert spaces

1. (a) Give the definition of inner product $\langle\ ,\ \rangle$ in a vector space $\mathbb{H}$. Give also a precise definition for $(\mathbb{H}, \langle\ ,\ \rangle)$ to be a Hilbert space and define the norm of an element $x \in \mathbb{H}$.

(b) Let $(\mathbb{H}, \langle\ ,\ \rangle)$ be a pre-Hilbert space. Prove that for $x, y \in \mathbb{H}$,

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2 .$$

(c) Let $\mathbb{H}$ be a vector space with a norm that satisfies the equation in part (b). Define

$$B(x,y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) .$$

Prove that

$$2B(x,y) = B(x+z,y) + B(x-z,y).$$

2. Let $(\mathbb{H}, \langle\ ,\ \rangle)$ be a Hilbert space and $\{e_n\}_{n=1}^{\infty}$ a sequence of elements of $\mathbb{H}$ or norm 1, that is $\|e_n\| = 1$, for all $n \in \mathbb{N}$. Suppose that for all $x \in \mathbb{H}$,

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = \|x\|^2.$$

(a) Prove that $\{e_n\}_{n=1}^{\infty}$ is and orthonormal system of $\mathbb{H}$.

(b) For $x \in \mathbb{H}$ and $N \in \mathbb{N}$ define $x_N = \sum_{n=1}^{N} \langle x, e_n\rangle e_n$. Prove that $\{x_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $\mathbb{H}$. *Hint: Use that for $x \in \mathbb{H}$, $\|x\| = \sup_{\|y\|=1} |\langle x, y\rangle|$.*

3. In $L^2([-1,1])$ consider the inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}dx ,$$

and the functions $p_0(x) = 1, p_1(x) = x, p_2(x) = x^2$ defined in $[-1,1]$. Find real numbers a and b such that the function $g(x) = ap_0(x) + bp_1(x) + p_2(x)$ is orthogonal to $p_0(x)$ and $p_1(x)$.

Fourier coefficients

4. (a) Give the definition of Fourier coefficients and Fourier series of a function $f \in L_p^2([-1/2, 1/2])$.

(b) Compute the Fourier coefficients of the function $f \in L_p^2([-1/2, 1/2])$ given by

$$f(x) = \begin{cases} 1 & \text{if } -\frac{1}{2} \le x \le 0 \\ 0 & \text{if } 0 < x \le \frac{1}{2} . \end{cases}$$

and extended to $\mathbb{R}$ with period 1. Start computing $\widehat{f}(0)$.

(c) Compute the Fourier coefficients of the function $g \in L^2_p([-1/2, 1/2])$ given by

$$g(x) = \begin{cases} -x & \text{if } -\frac{1}{2} \leq x \leq 0 \\ x & \text{if } 0 < x \leq \frac{1}{2} \,. \end{cases}$$

and extended to $\mathbb{R}$ with period 1. Start computing $\widehat{g}(0)$.

Fourier transform

5. (a) Prove that if $f \in L^1(\mathbb{R})$ and $xf \in L^1(\mathbb{R})$, then

$$\frac{d}{dw}\mathcal{F}(f)(w) = \mathcal{F}(-2\pi i x f)(w)\,.$$

Let $f(x) = e^{-4\pi^2 x^2}$ that verifies $\displaystyle\int_{\mathbb{R}} e^{-4\pi^2 x^2}\, dx = \frac{1}{2\sqrt{\pi}}$.

(b) Prove that $xf \in L^1(\mathbb{R})$ and, using part (a), show that

$$\frac{d}{dw}\mathcal{F}(f)(w) = -\frac{1}{2} w \mathcal{F}(f)(w)\,.$$

Hint: integrate by parts.

(c) Deduce from (b) that $\mathcal{F}(f)(w) = \dfrac{1}{2\sqrt{\pi}} e^{-w^2/2}$.

6. (a) Define the convolution, $f * g$, of two functions f and g define on the real line.

Let $f = \chi_{[0,1]}$ and $g = \chi_{[1,2]}$.

(b) Compute the $\mathcal{F}f$ and $\mathcal{F}g$, the Fourier transforms of f and g.

(c) Compute the convolution, $f * g$, of f and g, and also, $\mathcal{F}(f * g)$, the Fourier transform of the convolution $f * g$.

Orthonormal systems and basis

7. By Corollary 1.6.5, the set $\{e_n(x) = e^{2\pi i n x} : n \in \mathbb{Z}\}$ is complete in $L^2_p([0, 1])$. Hence, by Theorem 1.4.9, for all $f \in L^2_p([0, 1])$,

$$\|f\|^2_{L^2_p([0,1])} = \sum_{n=-\infty}^{\infty} |\langle f, e_n\rangle|^2_{L^2_p([0,1])} \qquad \text{(Plancherel identity)}.$$

Prove that the set

$$\{u_n(x) = \frac{1}{\sqrt{N}} e^{2\pi i \frac{n}{T} x} : n \in \mathbb{Z}\}$$

is complete in $L_p^2([a,b])$, where $T = b - a$. *Hint: for any* $g \in L_p^2([a,b])$ *define* $f(x) = g(a+Tx)$*; prove that* $f \in L_p^2([0,1])$ *and use Plancherel identity for* f *to prove Plancherel identity for* g*, that is,*

$$\|g\|^2_{L_p^2([a,b])} = \sum_{n=-\infty}^{\infty} |\langle g, u_n\rangle|^2_{L_p^2([a,b])} \,.$$

8. It was proved in class (Proposition 3.1.3) that the set $\left\{ C_n(x) = \sqrt{2}\cos\dfrac{\pi(2k+1)x}{2} \right\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2([0,1])$.

(a) Write Plancherel identity for this orthonormal basis.

Let $M \in \mathbb{N}, M > 1$ and consider the set

$$\left\{ U_n(x) = \sqrt{\frac{2}{M}}\cos\frac{\pi(2k+1)x}{2M} \right\}_{k=1}^{\infty}.$$

(b) Prove that $\{U_n(x)\}_{k=1}^{\infty}$ is an orthonormal system of $L^2([0,M])$.

(c) Show that the system $\{U_n(x)\}_{k=1}^{\infty}$ is complete in $L^2([0,M])$. *Hint: use part (a)*

9. (a) Write the statement of the Whittaker-Shannon sampling theorem when $f \in L^2(\mathbb{R})$ and supp $\mathcal{F}f \subset [-\frac{1}{2}, \frac{1}{2}]$.

(b) Let $f \in L^2(\mathbb{R})$ be given by

$$\mathcal{F}f(w) = \begin{cases} 2w+1 & \text{if } -\frac{1}{2} \le x \le 0 \\ -2w+1 & \text{if } \ 0 < x \le \frac{1}{2} \\ 0 & \text{otherwise}. \end{cases}$$

Compute f using $\mathcal{F}^{-1}$.

(c) Write Whittaker-Shannon sampling theorem for the function f given in (b).

Discrete transformations and bases

10. Let $f = (f(n))_{n=0}^{N-1}$ be a discrete signal of size N.

(a) Give the definition of the discrete Fourier transform (DFT) of f.

(b) Describe the fast Fourier transform (FFT) algorithm to compute DFT.

(c) Let $f = (f(0), f(1), f(2), f(3)) = (1, 0, 1, 0)$ be a discrete signal of size 4. Compute DFT with the definition given in part (a).

(d) For f as in part (c). Compute DFT using the FFT algorithm described in part (b).

11. (a) Describe the discrete cosine-I basis (DC-I) for the space of signals of size N and give the definition of the discrete cosine-I transform (DCT-I).

(b) Compute de DCT-I of the signal $f = (f(0), f(1), f(2), f(3)) = (1, 0, 1, 0)$.

12. Let

$$X = \text{MASTERMATHEMATICS}$$

be a text made up with the letters of the alphabet

$$\mathcal{S} = \{\text{ A , E , I, C, H, M , R, S , T}\}$$

(a) Compute $\mathcal{E}(X)$, the entropy of X.

(b) Find a Huffman code for X.

Orthonormal wavelets

13. Let $\varphi(x) = \frac{1}{3}\chi_{[0,3)}$ be a function defined on $\mathbb{R}$.

(a) Show that $\mathcal{F}f(0) = 1$ and $\mathcal{F}\varphi(w) = e^{-3\pi i w}\frac{\sin 3\pi w}{3\pi w}$ if $w \neq 0$.

(b) Define $h(w)$ by the formula $\mathcal{F}\varphi(2w) = h(w)\mathcal{F}\varphi(w)$. Show that $h(w) = \dfrac{1 + e^{-6\pi i w}}{2}$ and that $|h(w)|^2 + |h(w + 1/2)|^2 = 1$ for all $w \in \mathbb{R}$.

(c) Define

$$h[k] = \langle \frac{1}{2}\varphi(\frac{x}{2}), T_{3k}\varphi\rangle = \int_{-\infty}^{\infty} \frac{1}{2}\varphi(\frac{x}{2})\overline{\varphi(x - 3k)}dx.$$

Show that $h[0] = \dfrac{1}{6} = h[1]$, and $h[k] = 0$ if $k \neq 0, 1$.

14. (a) Give the definition of Multiresolution Analysis (MRA) $\left(\{V_j\}_{j\in\mathbb{Z}}, \varphi\right)$.

(b) Consider $\varphi_{j,k}(x) = D_{2^j}T_k\varphi(x)$. Find $\mathcal{F}(\varphi_{j,k})(w)$.

(c) Prove that $\{\varphi_{j,k} : k \in \mathbb{Z}\}$ is and orthonormal basis of V_j using that $\{\varphi_{0,k} : k \in \mathbb{Z}\}$ is and orthonormal basis of V_0.

15. (a) Explain the way S. Mallat produced orthonormal wavelets in $L^2(\mathbb{R})$ starting with a Multiresolution Analysis (MRA) $\left(\{V_j\}_{j\in\mathbb{Z}}, \varphi\right)$.

(b) Let $\varphi(x) = \chi_{[2,3]}$ be the scaling function fo an MRA. Find the low-pass filter $h(w)$ associated to φ.

(c) Find the high-pass filter $g(w)$ associated to φ given by $g(w) = e^{-2\pi i w}\overline{h(w + \frac{1}{2})}$.

(d) Find the orthonormal wavelet ψ associated to the scaling function φ an the high-pass filter $g(w)$.