Agostinho Neto University – College of Sciences
Master in Mathematics and Applications
Wavelets and Signal Processing - 2021 Homework

1. Prove that if M is a closed subspace of a Hilbert space $(\mathcal{H}, \langle\ \rangle)$, then $(M^\perp)^\perp = M$.

2. Prove that, for an interval $[a, b]$ of length T, the set of functions $\{e_n : n \in \mathbb{Z}\}$ given by

$$e_n(x) = \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} x}, \quad x \in \mathbb{R},$$

is an orthornormal system of $L^2_p([a, b])$ with inner product given by $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$.

3. Prove that, for an interval $[a, b]$ of length T, the set of functions $\{e_n : n \in \mathbb{Z}\}$ given by

$$e_n(x) = \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} x}, \quad x \in \mathbb{R},$$

is complete in $L^2_p([a, b])$ with inner product given by $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$. (Hint: Prove that the Plancherel identity holds for $f \in L^2_p([a, b])$ using that it holds for the space $L^2_p([0, 1])$ for an appropriate exponential basis.)

4. (a) For $f = \chi_{[a,b]}$ compute $\mathcal{F}f(w)$ and prove that $\lim_{|w|\to\infty} \mathcal{F}f(w) = 0$.

(b) If $f = \sum_{i=1}^{\infty} \alpha_i \chi_{[a_i,b_i]}$ is a simple function prove that $\lim_{|w|\to\infty} \mathcal{F}f(w) = 0$.

5. Prove that if $f, g \in L^2(\mathbb{R})$, then $f * g \in L^\infty(\mathbb{R})$ and $\|f * g\|_\infty \leq \|f\|_2\|g\|_2$. (Hint: Use Cauchy-Schwarz inequality).

6. Prove that if $f \in L^2(\mathbb{R} \times \mathbb{R})$ and $supp\, \mathcal{F}f \subset [-\frac{T_1}{2}, \frac{T_1}{2}] \times [-\frac{T_1}{2}, \frac{T_1}{2}]$, $T_1, T_2 > 0$, then

$$f(x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} f(\frac{k_1}{T_1}, \frac{k_2}{T_2}) \frac{\sin \pi(T_1 x - k_1)}{\pi(T_1 x - k_1)} \frac{\sin \pi(T_2 x - k_2)}{\pi(T_2 x - k_2)}$$

with convergence in $L^2(\mathbb{R} \times \mathbb{R})$ and uniformly on $\mathbb{R} \times \mathbb{R}$.

7. (a) Build a Huffman code $\mathcal{C}$ for the following text

$$X = 13524135352413513524111$$

made up with the symbols of $S = \{1, 2, 3, 4, 5\}$.

(b) Compute $\mathcal{E}(X)$ and $M_X(\mathcal{C})$.

8. Let $(\{V_j :\}_{j\in\mathbb{Z}}, \varphi)$ be a Multiresolution Analysis. Let W_0 be the orthogonal complement of V_0 in V_1, that is $V_0 \oplus W_0 = V_1$.. For $j \in \mathbb{Z}, j \neq 0$, define

$$W_j = \{D_{2^j} f : f \in W_0\}.$$

Show that $V_j \oplus W_j = V_{j+1}$ for all $j \in \mathbb{Z}$.

9. Let $h(w)$ and $g(w)$ be two functions in $L^2_p([0,1])$ of the form

$$h(w) = \sum_{k=-\infty}^{\infty} h[k]e^{-2\pi ikw}, \qquad g(w) = \sum_{k=-\infty}^{\infty} g[k]e^{-2\pi ikw}.$$

If we have the relation $g(w) = e^{-2\pi iw}\overline{h(w+\frac{1}{2})}$, prove that

$$g[k] = \overline{h[1-k]}(-1)^k\,.$$

10. With computations similar to the ones of Proposition 4.5.1, prove that

$$d_{j-1} = \sqrt{2}\sum_{k=-\infty}^{\infty} \overline{g(x)[k-2p]}c_j[k],$$

where $\{g[k]\}_{k\in\mathbb{Z}}$ are the coefficients of the high pass filter.