22/07/2021

(1)

Proposición 4.4.4 $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$ MRA; $\psi \in L^2(\mathbb{R})$ y $W = \overline{\text{span}\{T_k\psi : k\in\mathbb{Z}\}}$. Entonces

$$V_0 \perp W \iff g(\omega)\overline{h(\omega)} + g(\omega+\tfrac{1}{2})\overline{h(\omega+\tfrac{1}{2})} = 0 \text{ c.t. } \omega\in\mathbb{R}$$

D/ Por la prop 4.4.2.

$$V_0 \perp W \iff \sum_{k=-\infty}^{\infty} \mathcal{F}\varphi(\omega+k)\overline{\mathcal{F}\psi(\omega+k)} = 0 \text{ c.t. } \omega\in\mathbb{R} \iff$$

$$\iff \sum_{k=-\infty}^{\infty} \mathcal{F}\varphi(2\omega+k)\overline{\mathcal{F}\psi(2\omega+k)} = 0 \text{ c.t. } \omega\in\mathbb{R}$$

$$\iff \sum_{k=-\infty}^{\infty} \mathcal{F}\varphi(2(\omega+\tfrac{k}{2}))\overline{\mathcal{F}\psi(2(\omega+\tfrac{k}{2}))} = 0 \text{ c.t. } \omega\in\mathbb{R}$$

Sabemos que $\mathcal{F}\varphi(2\omega) = h(\omega)\mathcal{F}\varphi(\omega)$ (por (4.4.4)) y

$\mathcal{F}\psi(2\omega) = g(\omega)\mathcal{F}\varphi(\omega)$ (por (4.4.9))

$$0 = \sum_{k=-\infty}^{\infty} h(\omega+\tfrac{k}{2})\mathcal{F}\varphi(\omega+\tfrac{k}{2})\overline{g(\omega+\tfrac{k}{2})\mathcal{F}\varphi(\omega+\tfrac{k}{2})}$$

pares / impares $k=2l+1$

$k=2l$

$$= \sum_{l=-\infty}^{\infty} h(\omega+l)\overline{g(\omega+l)}\,|\mathcal{F}\varphi(\omega+l)|^2 + \sum_{l=-\infty}^{\infty} h(\omega+l+\tfrac{1}{2})\overline{g(\omega+l+\tfrac{1}{2})}\,|\mathcal{F}\varphi(\omega+\tfrac{1}{2}+l)|^2$$

(h y g son 1-periódicas)

$$= \sum_{l=-\infty}^{\infty} h(\omega)\overline{g(\omega)}\,|\mathcal{F}\varphi(\omega+l)|^2 + \sum_{l=-\infty}^{\infty} h(\omega+\tfrac{1}{2})\overline{g(\omega+\tfrac{1}{2})}\,|\mathcal{F}\varphi(\omega+\tfrac{1}{2}+l)|^2$$

$$= h(\omega)\overline{g(\omega)}\underbrace{\left(\sum_{l=-\infty}^{\infty}|\mathcal{F}\varphi(\omega+l)|^2\right)}_{1} + h(\omega+\tfrac{1}{2})\overline{g(\omega+\tfrac{1}{2})}\underbrace{\left(\sum_{l=-\infty}^{\infty}|\mathcal{F}\varphi(\omega+\tfrac{1}{2}+l)|^2\right)}_{1}$$

Prop 4.3.2

$$\implies h(\omega)\overline{g(\omega)} + h(\omega+\tfrac{1}{2})\overline{g(\omega+\tfrac{1}{2})}.$$

$\blacksquare$

NOTA Si $\mathcal{F}\varphi(0)=1$, de $\mathcal{F}\varphi(2\omega)=h(\omega)\mathcal{F}\varphi(\omega)$, se deduce $h(0)=1$. Como $|h(\omega)|^2+|h(\omega+\frac{1}{2})|^2=1$, tenemos $h(\frac{1}{2})=0$. Por la Prop 4.4.4, $g(0)\overline{h(0)}+g(\frac{1}{2})\overline{h(\frac{1}{2})}=0$ implica $g(0)=0$. Como $\mathcal{F}\psi(2\omega)=g(\omega)\mathcal{F}\varphi(\omega)$, se tiene $\mathcal{F}\psi(0)=0$

$$\Rightarrow \int_{-\infty}^{\infty}\psi(x)\,dx=0$$

Prop 4.4.5 $\{T_k\psi : k\in\mathbb{Z}\}$ es base o.n. de W_0 $\Longleftrightarrow$

(i) $|g(\omega)|^2+|g(\omega+\frac{1}{2})|^2=1$ c.t. $\omega\in\mathbb{R}$

(ii) $g(\omega)\overline{h(\omega)}+g(\omega+\frac{1}{2})\overline{h(\omega+\frac{1}{2})}=0$ c.t. $\omega\in\mathbb{R}$

D/ $\Rightarrow$ (i) es la Prop 4.4.3

(ii) es la Prop 4.4.4.

$\Leftarrow$) Ver Cap 2 de [HW]

Teorema 4.4.6 (S. Mallat, 1989)

$(\{V_j\}_{j\in\mathbb{Z}},\varphi)$ MRA de $L^2(\mathbb{R})$; sea $h(\omega)$ el filtro de paso bajo del MRA. Definir

$$g(\omega)=e^{-2\pi i\omega}\,\overline{h(\omega+\tfrac{1}{2})}\,\nu(2\omega)$$

donde $\nu(\omega)$ es 1-periódica y $|\nu(\omega)|=1$ c.t. $\omega\in\mathbb{R}$.

Sea ψ la función dada por

$$\mathcal{F}\psi(\omega)=g\left(\frac{\omega}{2}\right)\mathcal{F}\varphi\left(\frac{\omega}{2}\right), \qquad (4.4.10)$$

El conjunto $\{\psi_{j,k} : j\in\mathbb{Z}, k\in\mathbb{Z}\}$ es base o.n. de $L^2(\mathbb{R})$. Por tanto, ψ es una ondícula o.n.

$$L^2(\mathbb{R})=\bigoplus_{j=-\infty}^{\infty}W_j.$$

D/ Basta probar (i) y (ii) de la Prop 4.4.5

(i) $|g(\omega)|^2+|g(\omega+\frac{1}{2})|^2 = |e^{-2\pi i\omega}\overline{h(\omega+\frac{1}{2})}\,\vartheta(2\omega)|^2 +$

$+|e^{-2\pi i(\omega+\frac{1}{2})}\overline{h(\omega+1)}\,\vartheta(2\omega+1)|^2 = |h(\omega+\frac{1}{2})|^2+|h(\omega)|^2 = 1$

por la Prop. 4.4.1.

(ii) $g(\omega)\overline{h(\omega)} + g(\omega+\frac{1}{2})\overline{h(\omega+\frac{1}{2})} =$

$= e^{-2\pi i\omega}\overline{h(\omega+\frac{1}{2})}\,\vartheta(2\omega)\,\overline{h(\omega)} + e^{-2\pi i(\omega+\frac{1}{2})}\overline{h(\omega)}\,\vartheta(2\omega+\cancel{1})\,\overline{h(\omega+\frac{1}{2})}$

$\vartheta(2\omega)$

$= e^{-2\pi i\omega}\overline{h(\omega+\frac{1}{2})}\,\vartheta(2\omega)\,\overline{h(\omega)}\left[1+e^{-\pi i}\right] = 0$

$\boxed{e^{-\pi i}=-1 \quad \text{Euler}}$

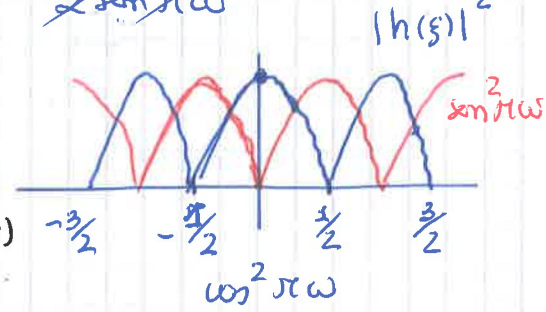
Ejercicio 4.4.3 Sea $\varphi(x)=\chi_{[0,1]}(x)$. Hallar el filtro de paso bajo $h(\omega)$ y sus coeficientes.

S/ Solución 1. El filtro $h(\omega)$ satisface $\mathcal{F}\varphi(2\omega)=h(\omega)\mathcal{F}\varphi(\omega)$ (ver (4.4.4)). Calculamos $\mathcal{F}\varphi(\omega)=e^{-\pi i\omega}\frac{\operatorname{sen}\pi\omega}{\pi\omega}$ (ver Ejerc. 4.3.3)

$$e^{-\pi i(2\omega)}\frac{\operatorname{sen}\pi(2\omega)}{\pi(2\omega)} = h(\omega)\,e^{-\pi i\omega}\frac{\operatorname{sen}\pi\omega}{\pi\omega}$$

$$h(\omega) = e^{-\pi i\omega}\frac{\operatorname{sen}2\pi\omega}{2\operatorname{sen}\pi\omega} = e^{-\pi i\omega}\frac{\cancel{2}\,\cancel{\operatorname{sen}\pi\omega}\cos\pi\omega}{\cancel{2}\,\cancel{\operatorname{sen}\pi\omega}}$$

$$\boxed{h(\omega) = e^{-\pi i\omega}\cos\pi\omega}$$



Obs

$|h(\omega)|^2+|h(\omega+\frac{1}{2})|^2 = \cos^2\pi\omega + \cos^2\pi(\omega+\frac{1}{2})$

$= \cos^2\pi\omega + \operatorname{sen}^2\pi\omega = 1$

$h(\omega)=\sum_{k=-\infty}^{\infty} h[k]\,e^{-2\pi i k\omega}$; $h(\omega) = e^{-\pi i\omega}\cos\pi\omega = e^{-\pi i\omega}\frac{e^{\pi i\omega}+e^{-\pi i\omega}}{2}$

$= \frac{1}{2}+\frac{1}{2}e^{-2\pi i\omega} \Rightarrow \left.\begin{array}{l} h[0]=1/2 \\ h[1]=1/2 \end{array}\right\}$ El resto $h[k]=0$

<u>Solución 2</u> Usar la formula

$$h[k] = \langle \tfrac{1}{2}\varphi(\tfrac{x}{2}), T_k\varphi\rangle_2 = \frac{1}{2}\int_{-\infty}^{\infty} \varphi(\tfrac{x}{2})\overline{\varphi(x-k)}\,dx$$

probada en (4.4.2)

$$h[0] = \frac{1}{2}\int_{-\infty}^{\infty} \chi_{[0,1]}(x/2)\,\chi_{[0,1]}(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \chi_{[0,2]}(x)\chi_{[0,1]}(x)\,dx$$

$0 \leq \frac{x}{2} \leq 1 \Leftrightarrow 0 \leq x \leq 2$

$$= \frac{1}{2}\int_0^1 1\,dx = \frac{1}{2}$$

$$h[1] = \frac{1}{2}\int_{-\infty}^{\infty} \chi_{[0,1]}(x/2)\,\chi_{[0,1]}(x-1)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \chi_{[0,2]}(x)\chi_{[1,2]}(x)\,dx$$

$0 \leq x-1 \leq 1$

$$= \frac{1}{2}\int_1^2 1\,dx = \frac{1}{2}$$ $1 \leq x \leq 2$

Si $k \neq 0$ y $k \neq 1$

$$h[k] = \frac{1}{2}\int_{-\infty}^{\infty} \chi_{[0,1]}(x/2)\,\chi_{[0,1]}(x-k)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \chi_{[0,2]}(x)\,\chi_{[k,k+1]}(x)\,dx = 0$$

$0 \leq x-k \leq 1$

$k \leq x \leq 1+k$

porque $|[0,2] \cap [k, k+1]| = 0$ cuando $k \neq 0, 1$

Con estos coeficientes

$$h(\omega) = \sum_{k=-\infty}^{\infty} h[k]e^{-2\pi i k\omega} = \frac{1}{2} + \frac{1}{2}e^{-2\pi i\omega} = e^{-\pi i\omega}\,\frac{e^{\pi i\omega}+e^{-\pi i\omega}}{2}$$

$$= e^{-\pi i\omega}\cos\pi\omega.$$

Ejercicio 4.4.4 Usar el Teor. de S. Mallat (Teor 4.4.6) para hallar la ondícula ψ asociada al MRA de Haar ($\varphi = \chi_{[0,1]}$) cuando se toma $\nu(\omega) = 1$ para el filtro de paso alto

S/ Por el ejercicio anterior $h(\omega) = \frac{1}{2} + \frac{1}{2} e^{-2\pi i \omega}$. Para hallar $g(\omega)$:

$$g(\omega) = e^{-2\pi i \omega} \overline{h(\omega + \tfrac{1}{2})} = e^{-2\pi i \omega} \overline{\left(\frac{1}{2} + \frac{1}{2} e^{-2\pi i (\omega + \frac{1}{2})} \right)}$$

$$= \frac{1}{2} e^{-2\pi i \omega} + \frac{1}{2} e^{\pi i} = -\frac{1}{2} + \frac{1}{2} e^{-2\pi i \omega} =$$

$$= e^{-\pi i \omega} \frac{-e^{\pi i \omega} + e^{-\pi i \omega}}{2} = e^{-\pi i \omega} i \operatorname{sen}(-\pi \omega) = -i e^{-\pi i \omega} \operatorname{sen} \pi \omega$$

La ondícula ψ satisface $\mathcal{F}\psi(2\omega) = g(\omega) \mathcal{F}\varphi(\omega)$

$$\mathcal{F}\psi(\omega) = g(\tfrac{\omega}{2}) \mathcal{F}\varphi(\tfrac{\omega}{2}) = \left(-\frac{1}{2} + \frac{1}{2} e^{-2\pi i \omega/2}\right) e^{-\pi i \frac{\omega}{2}} \frac{\operatorname{sen} \pi \omega/2}{\pi \omega/2}$$

$$= -\frac{1}{2} \underbrace{e^{-\pi i \omega/2} \frac{\operatorname{sen} \pi \omega/2}{\pi \omega/2}}_{\mathcal{F}\varphi(\omega/2)} + \frac{1}{2} e^{-\pi i \omega} \underbrace{e^{-\pi i \omega/2} \frac{\operatorname{sen} \pi \omega/2}{\pi \omega/2}}_{\mathcal{F}\varphi(\omega/2)}$$

$$= -\frac{1}{2} \mathcal{F}\varphi(\tfrac{\omega}{2}) + \frac{1}{2} M_{-1} \mathcal{F}\varphi(\omega/2) =$$

$$= -\frac{1}{2} \mathcal{F}\varphi(\tfrac{\omega}{2}) + \frac{1}{2} \mathcal{F}(T_1\varphi)(\tfrac{\omega}{2})$$

- $$\mathcal{F}\varphi(\tfrac{\omega}{2}) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i \frac{\omega}{2} x} dx \overset{\frac{x}{2} = y}{=} \int_{-\infty}^{\infty} \varphi(2y) e^{-2\pi i \omega y} 2 dy$$

$$= \sqrt{2} \int_{-\infty}^{\infty} \underbrace{\sqrt{2}\varphi(2y)}_{D_2\varphi} e^{-2\pi i \omega y} dy = \sqrt{2} \mathcal{F}(D_2\varphi)(\omega)$$

- $$\mathcal{F}(T_1\varphi)(\tfrac{\omega}{2}) = \cdots\cdots = \sqrt{2} \mathcal{F}(D_2 T_1 \varphi)(\omega)$$

$$= -\frac{1}{2} (\sqrt{2} \mathcal{F}(D_2\varphi)(\omega)) + \frac{1}{2} \sqrt{2} \mathcal{F}(D_2 T_1 \varphi)(\omega)$$

$$= \mathcal{F}\left(-\frac{1}{\sqrt{2}} D_2\varphi + \frac{1}{\sqrt{2}} D_2 T_1 \varphi\right)(\omega)$$

$$\psi(x) = -\frac{1}{\sqrt{2}} D_2\varphi(x) + \frac{1}{\sqrt{2}} D_2 T_1 \varphi(x) =$$

$$= -\frac{1}{\sqrt{2}}\sqrt{2}\,\varphi(2x) + \frac{1}{\sqrt{2}}\sqrt{2}\,\varphi(2x-1)$$

$$= -\varphi(2x) + \varphi(2x-1)$$

$$= -\chi_{[0,1]}(2x) + \chi_{[0,1]}(2x-1)$$

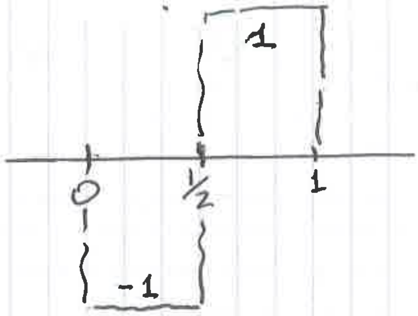
$0 \le 2x-1 \le 1$

$\frac{1}{2} \le x \le 1$

$0 \le 2x \le 1$

$0 \le x \le 1/2$

$$= -\chi_{[0,\frac{1}{2}]}(x) + \chi_{[1/2,1]}(x)$$



Ejercicio 4.4.5 (Para entregar 24/07) Sean $h(\omega)$ y $g(\omega)$ de la forma

$$h(\omega) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-2\pi i k\omega} \quad , \quad g(\omega) = \sum_{k=-\infty}^{\infty} g[k]\, e^{-2\pi i k\omega}.$$

Si se cumple que $g(\omega) = e^{-2\pi i\omega}\,\overline{h(\omega+\frac{1}{2})}$, probar que

$$g[k] = \overline{h[1-k]}\,(-1)^{1-k}$$

Ejercicio 4.4.6 (Ondícula de Shannon) Sea $\varphi \in L^2(\mathbb{R})$ t.g. $\mathcal{F}\varphi = \chi_{[-1/2,1/2]}$

(a) Hallar el filtro de paso bajo de φ

(b) Con $g(\omega) = e^{-2\pi i\omega}\,\overline{h(\omega+\frac{1}{2})}$, hallar $g(\omega)$, que es el filtro de paso alto

S/ (a) Usar $\mathcal{F}\varphi(2\omega) = h(\omega)\,\mathcal{F}\varphi(\omega)$ (ver (4.4.4))

$$\chi_{[-1/2,1/2]}(2\omega) = h(\omega)\,\chi_{[-1/2,1/2]}(\omega)$$

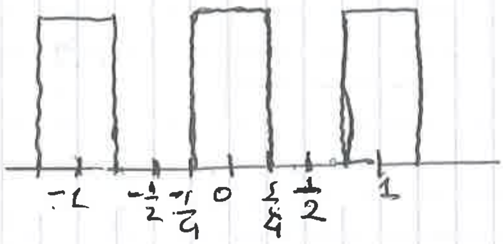
$$-\frac{1}{2} \le 2\omega \le \frac{1}{2} \iff -\frac{1}{4} \le \omega \le \frac{1}{4}$$

$$\chi_{[-\frac{1}{4},\frac{1}{4}]}(\omega) = h(\omega)\,\chi_{[-\frac{1}{2},\frac{1}{2}]}(\omega)$$

Si $\omega \in [-\frac{1}{2},\frac{1}{2}]$, $h(\omega) = \chi_{[-\frac{1}{4},\frac{1}{4}]}(\omega)$

Como $h(\omega)$ es 1-periódica

$$h(\omega) = \sum_{k=-\infty}^{\infty} \chi_{[-\frac{1}{4},\frac{1}{4}]}(\omega - k)$$



(b) $g(\omega) = e^{-2\pi i\omega}\, \overline{h(\omega+\frac{1}{2})} = e^{-2\pi i\omega} \sum_{k=-\infty}^{\infty} \chi_{[-\frac{1}{4},\frac{1}{4}]}(\omega + \frac{1}{2} - k)$

Como $g(\omega)$ es 1-periódica basta hallar sus valores en $[-\frac{1}{2},\frac{1}{2}]$.

$\boxed{-\frac{1}{2} \le \omega \le \frac{1}{2}}$

$-\frac{1}{4} \le \omega + \frac{1}{2} - k \le \frac{1}{4} \iff k - \frac{3}{4} \le \omega \le k - \frac{1}{4}$

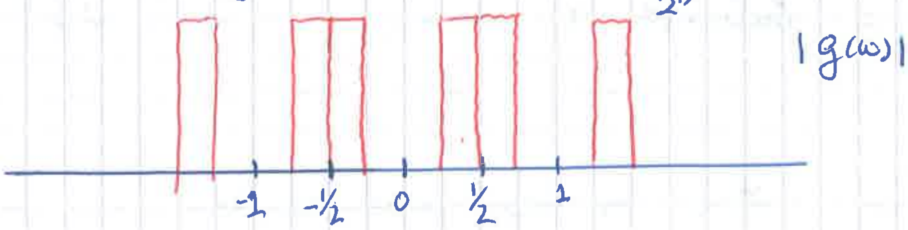
$\boxed{k=0}$ $-\frac{3}{4} \le \omega \le -\frac{1}{4}$ $\quad [-\frac{1}{2},\frac{1}{2}] \cap [-\frac{3}{4},-\frac{1}{4}] = [-\frac{1}{2},-\frac{1}{4}]$

$\boxed{k=1}$ $\frac{1}{4} \le \omega \le \frac{3}{4}$ $\quad [-\frac{1}{2},\frac{1}{2}] \cap [\frac{1}{4},\frac{3}{4}] = [\frac{1}{4},\frac{1}{2}]$

$\boxed{k=2}$ $\frac{5}{4} \le \omega \le \frac{7}{4}$ $\quad [-\frac{1}{2},\frac{1}{2}] \cap [\frac{5}{4},\frac{7}{4}] = \emptyset$

$\forall k \neq 0, 1$, la intersección es vacía

Si $\omega \in [-\frac{1}{2},\frac{1}{2}]$, $g(\omega) = e^{-2\pi i\omega}\left[\chi_{[-\frac{1}{2},-\frac{1}{4}]}(\omega) + \chi_{[\frac{1}{4},\frac{1}{2}]}(\omega)\right]$



4.5. TRANSFORMADA DE ONDÍCULAS

$(\{V_j\}_{j\in A}, \varphi)$ MRA

$$
\begin{array}{ccccccccc}
V_0 & \hookrightarrow & V_1 & \hookrightarrow & \cdots & V_{J-2} & \hookrightarrow & V_{J-1} & \hookrightarrow V_J \\
\oplus & \nearrow & \oplus & \nearrow & & \oplus & \nearrow & \oplus & \nearrow \\
W_0 & & W_1 & & & W_{J-2} & & W_{J-1} &
\end{array}
$$

$P_{V_j} f$ approxima a f en $L^2(\mathbb{R})$ cuando $j \to \infty$

$P_{W_{J-1}} f$ son los "detalles" que le falta a $P_{V_{J-1}} f$ para obtener $P_{V_J} f$

- $\{\varphi_{J,k} : k \in \mathbb{Z}\}$ base o.n. de V_J $\Rightarrow$

$$P_{V_J} f \overset{L^2(\mathbb{R})}{=\!=} \sum_{k=-\infty}^{\infty} c_J[k]\, \varphi_{J,k} \quad \text{con} \quad c_J[k] = \langle f, \varphi_{J,k} \rangle_2 \tag{4.5.1}$$

- $\{\psi_{J,k} : k \in \mathbb{Z}\}$ base o.n. de W_J $\Rightarrow$

$$P_{W_J} f \overset{L^2(\mathbb{R})}{=\!=} \sum_{k=-\infty}^{\infty} d_J[k]\, \psi_{J,k} \quad \text{con} \quad d_J[k] = \langle f, \psi_{J,k} \rangle_2 \tag{4.5.2}$$

Objetivos (1) Calcular $\{c_{J-1}[k]\}_{k\in\mathbb{Z}}$ y $\{d_{J-1}[k]\}_{k\in\mathbb{Z}}$ conociendo $\{c_J[k]\}_{k\in\mathbb{Z}}$

(2) Calcular $\{c_J[k]\}_{k\in\mathbb{Z}}$ conociendo $\{c_{J-1}[k]\}_{k\in\mathbb{Z}}$ y $\{d_{J-1}[k]\}_{k\in\mathbb{Z}}$.

Todo lo que necesitamos es usar los coeficientes de los filtros $\{h[k] : k\in\mathbb{Z}\}$ y $\{g[k] : k \in \mathbb{Z}\}$

Recordar que

$$\frac{1}{2}\varphi\left(\frac{x}{2}\right) = \sum_{k=-\infty}^{\infty} h[k]\,\varphi(x-k), \quad \text{con } h[k] = \left\langle \frac{1}{2}\varphi\left(\frac{x}{2}\right), T_k\varphi \right\rangle_2 \qquad (4.5.3)$$

Como $\varphi_{j-1,p} \in V_{j-1} \subset V_j$

$$\varphi_{j-1,p} \overset{L^2(\mathbb{R})}{=} \sum_{k=-\infty}^{\infty} \langle \varphi_{j-1,p}, \varphi_{j,k} \rangle \varphi_{j,k} \quad \text{con}$$

$$\langle \varphi_{j-1,p}, \varphi_{j,k} \rangle_2 = \int_{-\infty}^{\infty} 2^{\frac{j-1}{2}} \varphi(\underbrace{2^{j-1}x - p}_{\frac{1}{2}(2^j x - 2p)})\, 2^{j/2}\, \overline{\varphi(2^j x - k)}\, dx$$

$2^j x - 2p = y$

$$= \int_{-\infty}^{\infty} 2^{\frac{j-1}{2}} \varphi\left(\frac{y}{2}\right) 2^{j/2}\, \overline{\varphi(y + 2p - k)}\, \frac{dy}{2^j}$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \varphi\left(\frac{y}{2}\right) \overline{\varphi(y - (k-2p))} = \sqrt{2} \left\langle \frac{1}{2}\varphi\left(\frac{y}{2}\right), T_{k-2p}\varphi \right\rangle_2$$

$$\overset{(4.5.3)}{=} \sqrt{2}\, h[k-2p].$$

Por lo tanto

$$\varphi_{j-1,p} \overset{L^2(\mathbb{R})}{=} \sqrt{2} \sum_{k=-\infty}^{\infty} h[k-2p]\, \varphi_{j,k} \qquad (4.5.4)$$

Usando (4.5.1)

$$c_{j-1}[p] = \langle f, \varphi_{j-1,p} \rangle = \left\langle f, \sqrt{2} \sum_{k=-\infty}^{\infty} h[k-2p]\, \varphi_{j,k} \right\rangle$$

$$= \sqrt{2} \sum_{k=-\infty}^{\infty} \overline{h[k-2p]}\, \langle f, \varphi_{j,k} \rangle \overset{(4.5.1)}{=}$$

$$= \sqrt{2} \sum_{k=-\infty}^{\infty} \overline{h[k-2p]}\, c_j[k]$$

Prop 4.5.1. La transformada de ondículas cumple

$$c_{j-1}[p] = \sqrt{2} \sum_{k=-\infty}^{\infty} \overline{h[k-2p]}\, c_j[k]$$

Ejercicio 4.5.1 Con cálculos similares a los anteriores probar que

$$d_{j-1}[p] = \sqrt{2} \sum_{k=-\infty}^{\infty} \overline{g[k-2p]}\, c_j[k]$$

donde $\{g[k] : k \in \mathbb{Z}\}$ son los coeficientes del filtro de paso alto (para entregar 24-26/07)

Objetivo 2 $V_j = V_{j-1} \oplus W_{j-1} \Rightarrow \{\varphi_{j-1,k} : k \in \mathbb{Z}\} \cup \{\psi_{j-1,k} : k \in \mathbb{Z}\}$ es base o.n. de V_j. Como $\varphi_{j,p} \in V_j$

$$\varphi_{j,p} \overset{L^2(\mathbb{R})}{=} \sum_{k=-\infty}^{\infty} \langle \varphi_{j,p}, \varphi_{j-1,k} \rangle \varphi_{j-1,k} + \sum_{k=-\infty}^{\infty} \langle \varphi_{j,p}, \psi_{j-1,k} \rangle \psi_{j-1,k} \quad (4.5.5)$$

$$\langle \varphi_{j,p}, \varphi_{j-1,k} \rangle_2 \overset{(4.5.4)}{=} \langle \varphi_{j,p}, \sqrt{2} \sum_{\ell=-\infty}^{\infty} h[\ell-2k] \varphi_{j,\ell} \rangle_2$$

$$= \sqrt{2} \sum_{\ell=-\infty}^{\infty} \overline{h[\ell-2k]} \langle \varphi_{j,p}, \varphi_{j,\ell} \rangle_2 = \sqrt{2}\, \overline{h[p-2k]}$$

De manera similar

$$\langle \varphi_{j,p}, \psi_{j-1,k} \rangle = \sqrt{2}\, \overline{g[p-2k]}$$

Sustituir en (4.5.4):

$$\varphi_{j,p} \overset{L^2(\mathbb{R})}{=} \sqrt{2} \sum_{k=-\infty}^{\infty} \overline{h[p-2k]}\, \varphi_{j-1,k} + \sqrt{2} \sum_{k=-\infty}^{\infty} \overline{g[p-2k]}\, \psi_{j-1,k}$$

$$c_j[p] = \langle f, \varphi_{j,p} \rangle = \sqrt{2} \sum_{k=-\infty}^{\infty} h[p-2k] \underbrace{\langle f, \varphi_{j-1,k} \rangle}_{c_{j-1}[k]}$$

$$+ \sqrt{2} \sum_{\ell=-\infty}^{\infty} g[p-2k] \underbrace{\langle f, \psi_{j-1,k} \rangle}_{d_{j-1}[k]}$$

Prop. 4.5.2 Transformada inversa de ondículas:

$$c_j[p] = \sqrt{2}\sum_{k=-\infty}^{\infty} h[p-2k]\, c_{j-1}[k] + \sqrt{2}\sum_{k=-\infty}^{\infty} g[p-2k]\, d_{j-1}[k]$$

Ejercicio 4.5.2 Considerar el filtro del MRA de Haar (Ej 4.4.3) que es $h(\omega) = \frac{1}{2} + \frac{1}{2}e^{-2\pi i\omega}$. En el Ej. 4.4.4 hemos hallado $g(\omega) = -\frac{1}{2} + \frac{1}{2}e^{-2\pi i\omega}$

(a) Hallar la transformada de ondículas en este caso

(b) Sin usar la Prop 4.5.2, halla la transformada de ondículas inversa en este caso.

S/ (a) $c_{j-1}[p] = \sqrt{2}\sum_{k=-\infty}^{\infty} h[k-2p]\, c_j[k]$

$h[0] = \frac{1}{2}$, $h[1] = \frac{1}{2}$, si $k \neq 0, 1$, $h[k] = 0$

$$c_{j-1}[p] = \sqrt{2}\,[\underbrace{h[0]\, c_j[2p]}_{k=2p} + \underbrace{h[1]\, c_j[2p+1]}_{k=2p=1}]$$

$$= \sqrt{2}\,\frac{c_j[2p] + c_j[2p+1]}{2} \checkmark$$

$$d_{j-1}[p] = \sqrt{2}\sum_{k=-\infty}^{\infty} g[k-2p]\, c_j[k]$$

$g[0] = -\frac{1}{2}$, $g[1] = \frac{1}{2}$, si $k \neq 0, 1$, $g[k] = 0$

$$d_{j-1}[p] = \sqrt{2}\,[g[0]\, c_j[2p] + g(1)\, c_j[2p+1]]$$

$$= \sqrt{2}\,\frac{-c_j[2p] + c_j[2p+1]}{2} \checkmark$$

$$\left.\begin{aligned} 2c_{j-1}[p] &= \sqrt{2}\,c_j[2p] + \sqrt{2}\,c_j[2p+1] \\ 2d_{j-1}[p] &= -\sqrt{2}\,c_j[2p] + \sqrt{2}\,c_j[2p+1] \end{aligned}\right\}$$

$(-)\quad 2c_{j-1}[p] - 2d_{j-1}[p] = 2\sqrt{2}\,c_j[2p]$

$$c_j[2p] = \sqrt{2}\,\frac{c_{j-1}[p] - d_{j-1}[p]}{2}$$

$(+)\quad 2c_{j-1}[p] + 2d_{j-1}[p] = 2\sqrt{2}\,c_j[2p+1]$

$$c_j[2p+1] = \sqrt{2}\,\frac{c_{j-1}[p] + d_{j-1}[p]}{2}$$

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c_{j-1}

$\sqrt{2}$	3	7	11	15

$\sqrt{2}$	1	1	1	1

$c_j[k]$

$c_j[0]$	$c_j[1]$	$[2]$	$[3]$				$[7]$
2	4	6	8	10	12	14	16