20/07/2021

Función de Haar: $\psi(x)=\begin{cases}1 & \text{si } 0\le x\le 1/2\\ -1 & \text{si } \frac{1}{2}<x\le 1\\ 0 & \text{resto}\end{cases}$ (4,2.1)

Proposición 4.2.1. Con ψ como en (4.2.1), el conjunto $\{\psi_{j,k}: j,k\in\mathbb{Z}\}$ es un sistema o.n. en $L^2(\mathbb{R})$.

Ya sabemos que $\|\psi_{j,k}\|_2=1 \quad \forall j,k\in\mathbb{Z}$

La ortogonalidad se prueba en cuatro ejercicios

Ejercicio 4.2.2 $\forall k\in\mathbb{Z},\ k\neq 0,\ \langle\psi,\psi_{0,k}\rangle=0$

S/ $\operatorname{sop}\psi=[0,1]$; $\psi_{0,k}(x)=\psi(x-k)\neq 0 \iff 0\le x-k\le 1$ $\iff k\le x\le k+1$; $\operatorname{sop}\psi_{0,k}=[k,k+1]$. Entonces $|\operatorname{sop}\psi\cap\operatorname{sop}\psi_{0,k}|=|[0,1]\cap[k,k+1]|=0 \Rightarrow \langle\psi,\psi_{0,k}\rangle=0$

Ejercicio 4.2.3 $\forall j\in\mathbb{Z}$ y $k_1\neq k_2$, $\langle\psi_{j,k_1},\psi_{j,k_2}\rangle=0$

S/ $\langle\psi_{j,k_1},\psi_{j,k_2}\rangle=\int_{-\infty}^{\infty}2^{j/2}\psi(2^jx-k_1)\,2^{j/2}\overline{\psi(2^jx-k_2)}\,dx$

$2^jx-k_1=y$

$$=\int_{-\infty}^{\infty}\psi(y)\overline{\psi\left(2^j\left(\frac{y+k_1}{2^j}\right)-k_2\right)}\,dy=$$

$$=\int_{-\infty}^{\infty}\psi(y)\overline{\psi(y-(k_2-k_1))}\,dy=\langle\psi,\psi_{0,k_2-k_1}\rangle\overset{\text{Ejer 4.2.2}}{=}0$$

Ejercicio 4.2.4 Si $j>0$ y $k\in\mathbb{Z}$, $\langle\psi,\psi_{j,k}\rangle=0$

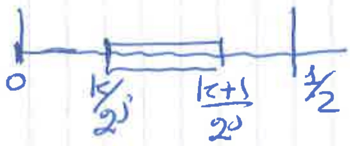
S/ $x\in\operatorname{sop}\psi_{j,k}\iff\psi_{j,k}(x)\neq 0\iff 2^{j/2}\psi(2^jx-k)\neq 0\iff$ $0\le 2^jx-k\le 1\iff k\le 2^jx\le k+1\iff\frac{k}{2^j}\le x\le\frac{k+1}{2^j}$

$$\operatorname{sop}\psi_{j,k}=I_{j,k}=\left[\frac{k}{2^j},\frac{k+1}{2^j}\right] \quad \text{(intervalos diádicos)}$$

- Si $k\ge 2^j$, $I_{j,k}\cap[0,1]=\emptyset \Rightarrow \langle\psi,\psi_{j,k}\rangle=0$

- Si $k<0$, $\frac{k+1}{2^j}\le 0$ y $I_{j,k}\cap[0,1]=\emptyset \Rightarrow \langle\psi,\psi_{j,k}\rangle=0$

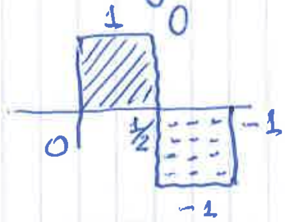
- $0\le k\le \frac{2^j}{2}-1$; $\frac{k+1}{2^j}\le \frac{2^j/2}{2^j}=\frac{1}{2}$



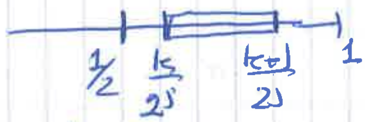
y $\frac{k}{2^j}\ge 0 \Rightarrow$ si $x\in I_{j,k}$, $\psi(x)=1$.

$$\langle\psi,\psi_{j,k}\rangle=\int_{k/2^j}^{\frac{k+1}{2^j}} 1\cdot 2^{j/2}\psi(2^jx-k)\,dx \;\underset{=}{\overset{2^jx-k=y}{}}\; \int_0^1 2^{-j/2}\psi(y)\,dy$$

$$=2^{-j/2}\int_0^1\psi(y)\,dy=0$$



- $\frac{2^j}{2}\le k\le 2^j-1$; $\frac{k}{2^j}\ge\frac{2^j/2}{2^j}=\frac{1}{2}$



$\frac{k+1}{2^j}\le\frac{2^j}{2^j}=1 \Rightarrow$ si $x\in I_{j,k}$, $\psi(x)=-1$

$$\langle\psi,\psi_{j,k}\rangle=\int_{k/2^j}^{\frac{k+1}{2^j}}(-1)\,2^{j/2}\psi(2^jx-k)\,dx \;\underset{=}{\overset{2^jx-k=y}{}}\; -\int_0^1 2^{-j/2}\psi(y)\,dy$$

$$=-2^{-j/2}\int_0^1\psi(y)\,dy=0.$$

Ejercicio 4.2.5 Probar que si $j_1\neq j_2$ ó $k_1\neq k_2$

$$\langle\psi_{j_1,k_1},\psi_{j_2,k_2}\rangle=0$$

S/ $j_2>j_1$

$$\langle\psi_{j_1,k_1},\psi_{j_2,k_2}\rangle=\int_{-\infty}^{\infty}2^{j_1/2}\psi(2^{j_1}x-k_1)\,2^{j_2/2}\psi(2^{j_2}x-k_2)\,dx \qquad 2^{j_1}x-k_1=y$$

$$=\int_{-\infty}^{\infty}2^{j_1/2}\psi(y)\,2^{j_2/2}\psi\left(2^{j_2}\left(\frac{y+k_1}{2^{j_1}}\right)-k_2\right)\cdot\frac{dy}{2^{j_1}}=$$

$$=\int_{-\infty}^{\infty}2^{\frac{j_2-j_1}{2}}\psi(y)\,\psi\left(2^{j_2-j_1}y-(k_2-2^{j_2-j_1}k_1)\right)dy$$

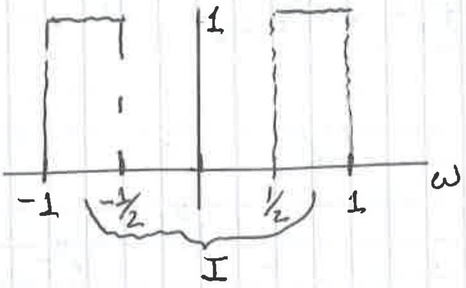
$$=\langle\psi,\psi_{j_2-j_1,\,k_2-2^{j_2-j_1}k_1}\rangle \overset{j_2-j_1>0}{=} 0 \qquad \blacksquare$$

Ej 1.2.4

4.2.2. LA ONDÍCULA DE SHANNON.

Sea $\psi \in L^2(\mathbb{R})$ definida por

$$\mathcal{F}\psi(\omega) = \chi_I(\omega) \qquad (4.2.2)$$



donde $I = [-1, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, 1]$.

$$\|\chi_I\|_2^2 = \int_{-1}^{-1/2} |1|^2 d\omega + \int_{1/2}^{1} |1|^2 d\omega$$

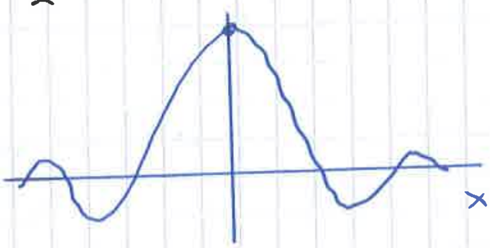
$$= \frac{1}{2} + \frac{1}{2} = 1$$

$\|\psi\|_2^2 \overset{(P)}{=} \|\mathcal{F}\psi\|_2^2 = \|\chi_I\|_2^2 = 1$ y por tanto

$$\|\psi_{j,k}\|_2^2 = \|D_{2^j} T_k \psi\|_2^2 = \|\psi\|_2^2 = 1.$$

Ejercicio 4.2.6. Prueba que

$$\psi(x) = \frac{\operatorname{sen}(2\pi x) - \operatorname{sen}(\pi x)}{\pi x}$$



S/

$$\psi(x) = \mathcal{F}^{-1}(\chi_I)(x) = \int_{-\infty}^{\infty} \chi_I(\omega)\, e^{2\pi i x \omega} d\omega$$

$$= \int_{-1}^{-1/2} e^{2\pi i x \omega} d\omega + \int_{1/2}^{1} e^{2\pi i x \omega} d\omega = \left[\frac{e^{2\pi i x \omega}}{2\pi i x}\right]_{-1}^{-1/2} + \left[\frac{e^{2\pi i x \omega}}{2\pi i x}\right]_{1/2}^{1}$$

$$= \frac{e^{-\pi i x} - e^{-2\pi i x} + e^{2\pi i x} - e^{\pi i x}}{2\pi i x} = \frac{e^{2\pi i x} - e^{-2\pi i x}}{2\pi i x} - \frac{e^{\pi i x} - e^{-\pi i x}}{2\pi i x}$$

$$= \frac{\operatorname{sen} 2\pi x - \operatorname{sen} \pi x}{\pi x}.$$

Proposición 4.2.2. Con ψ como en (4.2.2), el conjunto $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ es un s.o.n. en $L^2(\mathbb{R})$.

D/ Ya sabemos que $\|\psi_{j,k}\|_2 = 1$.

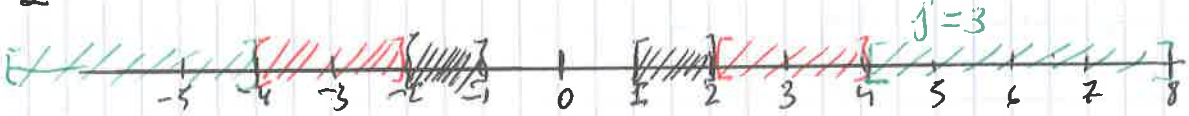
Si $j_1 \neq j_2$ ó $k_1 \neq k_2$

$$\langle \psi_{j_1,k_1}, \psi_{j_2,k_2}\rangle \underset{\text{Parseval}}{=} \langle \mathcal{F}(\psi_{j_1,k_1}), \mathcal{F}(\psi_{j_2,k_2})\rangle =$$

$$= \langle \mathcal{F}(D_{2^{j_1}}T_{k_1}\psi), \mathcal{F}(D_{2^{j_2}}T_{k_2}\psi)\rangle = \langle D_{2^{-j_1}}M_{-k_1}\chi_I, D_{2^{-j_2}}M_{-k_2}\chi_I\rangle$$

$$= \int_{-\infty}^{\infty} 2^{-j_1/2} e^{-2\pi i k_1 \frac{\omega}{2^{j_1}}}\chi_I\left(\frac{\omega}{2^{j_1}}\right) 2^{-j_2/2} e^{2\pi i k_2 \frac{\omega}{2^{j_2}}}\chi_I\left(\frac{\omega}{2^{j_2}}\right) d\omega \qquad (4.2.3)$$

- Si $j_1 \neq j_2$, $\chi_I\left(\frac{\omega}{2^{j_1}}\right) = 1 \iff \frac{\omega}{2^{j_1}} \in I \iff \frac{1}{2} \leq \frac{|\omega|}{2^{j_1}} \leq 1 \iff$
$\frac{2^{j_1}}{2} \leq |\omega| \leq 2^{j_1} \Rightarrow \text{sop}\, \chi_I\left(\frac{\omega}{2^{j_1}}\right) = \left\{\omega : \frac{2^{j_1}}{2} \leq |\omega| \leq 2^{j_1}\right\}$



Si $j_1 \neq j_2$, $\text{sop}\,\chi_I\left(\frac{\omega}{2^{j_1}}\right) \cap \text{sop}\,\chi_I\left(\frac{\omega}{2^{j_2}}\right) = \emptyset$ (c.t.p)

De (4.2.3), $\langle \psi_{j_1,k_2}, \psi_{j_2,k_2}\rangle = 0$

- Si $j_1 = j_2 = j$. Por (4.2.3) tenemos

$$\langle \psi_{j,k_1}, \psi_{j,k_2}\rangle = \int_{-\infty}^{\infty} 2^{-j} e^{-2\pi i (k_1-k_2)\frac{\omega}{2^j}} \chi_I\left(\frac{\omega}{2^j}\right) d\omega$$

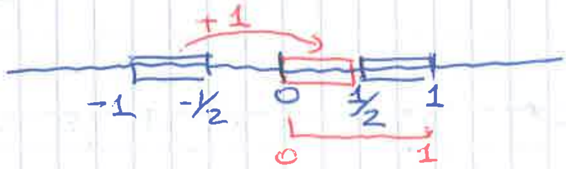
$$= \int_{-2^j}^{-2^j/2} 2^{-j} e^{-2\pi i(k_1-k_2)\frac{\omega}{2^j}} d\omega + \int_{2^j/2}^{2^j} 2^{-j} e^{-2\pi i(k_1-k_2)\frac{\omega}{2^j}} d\omega$$

$\frac{\omega}{2^j} = s - 1$ $\qquad$ $\frac{\omega}{2^j} = s$

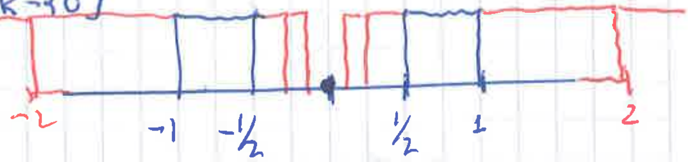
$$= \int_0^{1/2} e^{-2\pi i (k_1-k_2)s} ds + \int_{1/2}^{1} e^{-2\pi i(k_1-k_2)s} ds$$

$$= \int_0^1 e^{-2\pi i(k_1-k_2)s} ds = \begin{cases} 1 & \text{si } k_1 = k_2 \\ 0 & \text{si } k_1 \neq k_2 \end{cases}$$

(4.2.4) $\{ e^{2\pi i k\omega} : k \in \mathbb{Z} \}$ es b.o.n de $I = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$



(4.2.5) $$\sum_{j \in \mathbb{Z}} \chi_I (2^{-j} \omega) = 1 \quad \forall \omega \in \mathbb{R} - \{0\}$$



Teorema 4.2.3. Con ψ como en (4.2.2), el conjunto $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ es base o.n. de $L^2(\mathbb{R})$. Esta ψ es la ondícula de Shannon.

D/ Para probar que es completo, por el teorema 1.4.9 basta probar Plancherel: $\forall f \in L^2(\mathbb{R})$

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 = \|f\|^2 \qquad (4.2.6)$$

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 \overset{\text{Parseval}}{=} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle \mathcal{F} f, \mathcal{F} \psi_{j,k} \rangle|^2$$

$$= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle \mathcal{F} f, \mathcal{F} D_{2^j} T_k \psi \rangle|^2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle \mathcal{F} f, D_{2^{-j}} M_{-k} \mathcal{F} \psi \rangle|^2$$

$$= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left| \int_{-\infty}^{\infty} \mathcal{F} f(\omega)\, 2^{-j/2} e^{2\pi i k \frac{\omega}{2^j}} \overline{\mathcal{F}\psi \left(\frac{\omega}{2^j}\right)} \, d\omega \right|^2 \qquad \eta = \frac{\omega}{2^j}$$

$\eta =$ eta

$$= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left| \int_{-\infty}^{\infty} \mathcal{F} f(2^j \eta)\, 2^{j/2} e^{2\pi i k \eta} \overline{\mathcal{F}\psi(\eta)} \, d\eta \right|^2$$

$$= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^j \left| \int_I \mathcal{F} f(2^j \eta)\, e^{2\pi i k \eta} \, d\eta \right|^2 \qquad (\mathcal{F}\psi(\eta) = \chi_I)$$

$$\sum_{k\in\mathbb{Z}} \left| \int_I \mathcal{F}f(2^j\eta)\, e^{2\pi i k\eta} d\eta \right|^2 = \sum_{k\in\mathbb{Z}} \left| \langle \mathcal{F}f(2^j\cdot), e^{-2\pi i k\cdot} \rangle_{L^2(I)} \right|^2$$

Por (4.2.4) Plancherel $\displaystyle \int_I |\mathcal{F}f(2^j\eta)|^2 d\eta$. Sustituyendo,

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 = \sum_{j\in\mathbb{Z}} 2^j \int_I |\mathcal{F}f(2^j\eta)|^2 d\eta =$$

$$= \sum_{j\in\mathbb{Z}} 2^j \int_{-\infty}^{\infty} \chi_I(\eta) |\mathcal{F}f(2^j\eta)|^2 d\eta \qquad \underline{\underline{2^j\eta = \omega}}$$

$$= \sum_{j\in\mathbb{Z}} \int_{-\infty}^{\infty} \chi_I\left(\frac{\omega}{2^j}\right) |\mathcal{F}f(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \left(\sum_{j\in\mathbb{Z}} \chi_I(\omega/2^j) \right) |\mathcal{F}f(\omega)|^2 d\omega$$

$$\underset{(4.2.5)}{=} \int_{-\infty}^{\infty} |\mathcal{F}f(\omega)|^2 d\omega = \|\mathcal{F}f\|_2^2 \overset{(P)}{=} \|f\|_2^2 .$$

4.3. Análisis Multirresolución (MRA)

<u>Definición 4.3.1</u>. Un MRA es un conjunto $\{V_j\}_{j\in\mathbb{Z}}$ de subespacios cerrados de $L^2(\mathbb{R})$ tal que

(1) $V_j \subset V_{j+1} \quad \forall j \in \mathbb{Z}$ (2) $f\in V_j \Leftrightarrow D_2 f \in V_{j+1}$

(3) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$ (4) $\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R})$

(5) Existe $\varphi \in V_0$ (φ = phi) tal que $\{\varphi(x-k) : k\in\mathbb{Z}\}$ son una b.o.n. de V_0.

La función φ se llama función de escala del MRA.

<u>NOTA 1</u> (1) + (2) + (5) $\Rightarrow$ (3) [Cap 2, Teor 1.6 de HW]

<u>NOTA 2</u> Si se cumplen (1), (2) y (5) y $|\mathcal{F}\varphi(\omega)|$ es continuo en $\omega = 0$, son equivalentes

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}) \iff |\mathcal{F}\varphi(0)| \neq 0$$

[Cap 2, Teor 1.7 de HW].

4.3.1. Propiedades de la función de escala

<u>Prop. 4.3.2.</u> $g \in L^2(\mathbb{R})$. El conjunto $\{T_k g : k\in\mathbb{Z}\}$ es un sistema o.n. de $L^2(\mathbb{R}) \iff$

$$\sum_{k\in\mathbb{Z}} |\mathcal{F}g(\omega+k)|^2 = 1 \quad \text{c.t. } \omega \in [0,1)$$

D/
$$\langle g, T_k g\rangle_2 = \int_{-\infty}^{\infty} g(x)\,\overline{T_k g(x)}\,dx \overset{(P)}{=} \int_{-\infty}^{\infty} \mathcal{F}g(\omega)\,\overline{\mathcal{F}(T_k g)(\omega)}\,d\omega$$

$$= \int_{-\infty}^{\infty} \mathcal{F}g(\omega)\,\overline{M_{-k}\mathcal{F}g(\omega)}\,d\omega = \int_{-\infty}^{\infty} \mathcal{F}g(\omega)\, e^{2\pi i k\omega}\,\overline{\mathcal{F}g(\omega)}\,d\omega$$

$$= \int_{-\infty}^{\infty} |\mathcal{F}g(\omega)|^2 e^{2\pi i k \omega}\,d\omega = \sum_{\ell\in\mathbb{Z}} \int_{\ell}^{\ell+1} |\mathcal{F}g(\omega)|^2 e^{2\pi i k\omega}\,d\omega$$

$\omega = \eta + \ell$

$$= \sum_{\ell \in \mathbb{Z}} \int_0^1 |\mathcal{F}g(\eta+\ell)|^2 e^{2\pi i k(\eta+\ell)} d\eta$$

$\eta = \omega$ $\quad (e^{2\pi i k \eta} \cdot e^{2\pi i k \ell} = e^{2\pi i k \eta})$

$$= \sum_{\ell \in \mathbb{Z}} \int_0^1 |\mathcal{F}g(\omega+\ell)|^2 e^{2\pi i k \omega} d\omega$$

La suma anterior converge absolutamente porque

$$\sum_{\ell \in \mathbb{Z}} \int_0^1 |\mathcal{F}g(\omega+\ell)|^2 d\omega \overset{\omega+\ell=\eta}{=} \sum_{\ell \in \mathbb{Z}} \int_\ell^{\ell+1} |\mathcal{F}g(\eta)|^2 d\eta =$$

$$= \int_{-\infty}^{\infty} |\mathcal{F}g(\eta)|^2 d\eta = \|\mathcal{F}g\|_2^2 \overset{(P)}{=} \|g\|_2^2 < \infty. \qquad (3.3.1)$$

Por tanto

$$\langle g, T_k g \rangle = \int_0^1 \underbrace{\Big(\sum_{\ell \in \mathbb{Z}} |\mathcal{F}g(\omega+\ell)|^2\Big)}_{G(\omega)} e^{2\pi i k \omega} d\omega \qquad (3.3.2)$$

La función $G \in L^1([0,1])$ por (3.3.1) y es 1-periódica.

$\Rightarrow$) Si $\{T_k g : k \in \mathbb{Z}\}$ es s.o.n, entonces $\langle g, T_k g \rangle = \delta_{k,0}$. Por (3.3.2), $\int_0^1 G(\omega) e^{2\pi i k \omega} d\omega = \delta_{k,0} \Rightarrow$ $\hat{G}(0) = 1$ y $\hat{G}(k) = 0$ $\forall k \in \mathbb{Z}, k \neq 0$. Por el corolario 1.6.7, $G(\omega) = 1$ en casi todo punto.

$\Leftarrow$) Si $G(\omega) = 1$ en casi todo punto, por (2.3.2)

$$\langle g, T_k g \rangle = \int_0^1 1\, e^{2\pi i k \omega} d\omega = \begin{cases} 1 & \text{si } k = 0 \\ 0 & \text{si } k \neq 0 \end{cases} = \delta_{k,0}$$

Para $k_1 \neq k_2$

$x - k_1 = y$

$$\langle T_{k_1} g, T_{k_2} g \rangle = \int_{\mathbb{R}} g(x-k_1) \overline{g(x-k_2)} dx = \int_{\mathbb{R}} g(y) \overline{g(y-(k_2-k_1))} dy$$

$$= \langle g, T_{k_2-k_1} g \rangle = \begin{cases} 1 & \text{si } k_1 = k_2 \\ 0 & \text{si } k_1 \neq k_2 \end{cases}.$$

Ejercicio 4.3.1. $f, g \in L^2(\mathbb{R})$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$. Probar que

(a) $\langle T_k f, g \rangle = \langle f, T_{-k} g \rangle$ (b) $\langle D_{2^j} f, g \rangle = \langle f, D_{2^{-j}} g \rangle$

S/ (a) $\langle T_k f, g \rangle = \int_{-\infty}^{\infty} T_k f(x) \overline{g(x)}\, dx = \int_{-\infty}^{\infty} f(x-k) \overline{g(x)}\, dx$ $\underline{\underline{x-k=y}}$

$= \int_{-\infty}^{\infty} f(y) \overline{g(y+k)}\, dy = \int_{-\infty}^{\infty} f(y) \overline{T_{-k} g(y)}\, dy = \langle f, T_{-k} g \rangle$

(b) $\langle D_{2^j} f, g \rangle = \int_{-\infty}^{\infty} 2^{j/2} f(2^j x) \overline{g(x)}\, dx$ $\underline{\underline{2^j x = y}}$

$= \int_{-\infty}^{\infty} 2^{j/2} f(y) \overline{g\left(\frac{y}{2^j}\right)} \frac{dy}{2^j} = \int_{-\infty}^{\infty} f(y)\, \overline{2^{-j/2} g(2^{-j} y)}\, dy$

$= \int_{-\infty}^{\infty} f(y) \overline{D_{2^{-j}} g(y)}\, dy = \langle f, D_{2^{-j}} g \rangle$. ■

Prop. 4.3.3. φ es la función de escala de un MRA, $\{V_j\}_{j\in\mathbb{Z}}$. Definir

$$\varphi_{j,k}(x) = D_{2^j} T_k \varphi(x) = 2^{j/2} \varphi(2^j x - k), \quad j, k \in \mathbb{Z}.$$

El conjunto $\{\varphi_{j,k} : k \in \mathbb{Z}\}$ es base o.n. de V_j.

D/ $\varphi \in V_0 \Rightarrow T_k \varphi \in V_0 \overset{(2)}{\Rightarrow} D_{2^j} T_k \varphi \in V_j \Rightarrow \varphi_{j,k} \in V_j$

$\langle \varphi_{j,k_1}, \varphi_{j,k_2} \rangle = \int_{-\infty}^{\infty} 2^{j/2} \varphi(2^j x - k_1) \overline{2^{j/2} \varphi(2^j x - k_2)}\, dx$

$\underline{\underline{2^j x = y}}$ $\int_{-\infty}^{\infty} 2^{j/2} \varphi(y-k_1) \overline{2^{j/2} \varphi(y-k_2)} \frac{dy}{2^j} = \langle T_{k_1}\varphi, T_{k_2}\varphi \rangle$

$\overset{(5)}{=} \delta_{k_1,k_2}$. Probar que $\{\varphi_{j,k} : k \in \mathbb{Z}\}$ completo en V_j

$f \in V_j \overset{(2)}{\Longrightarrow} D_{2^{-j}} \in V_0$. Como $\{T_k \varphi : k \in \mathbb{Z}\}$ son b.o.n de V_0, se cumple Plancherel

$$\|D_{2^j}f\|_2^2 = \sum_{k\in\mathbb{Z}} |\langle D_{2^j}f, T_k\varphi\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle f, D_{2^j}T_k\varphi\rangle|^2$$

$$\Rightarrow \quad \|f\|_2^2 = \sum_{k\in\mathbb{Z}} |\langle f, \varphi_{j,k}\rangle|^2$$

que es Plancherel para $\{\varphi_{j,k} : k\in\mathbb{Z}\}$ y por tanto base o.n. por el teorema 1.4.9.

Sea $P_{V_j} : L^2(\mathbb{R}) \to V_j$ la proyección ortogonal de $L^2(\mathbb{R})$ sobre V_j. Las proyecciones ortogonales cumplen

$$P_{V_j} \circ P_{V_j} = P_{V_j} \quad \text{y} \quad P_{V_j}^* = P_{V_j}$$

Ejercicio 4.3.2 Sean $f, g \in L^2(\mathbb{R})$. Probar que

$$\langle P_{V_j} f, g\rangle = \langle f, P_{V_j} g\rangle$$

S/ $L^2(\mathbb{R}) = V_j \oplus V_j^\perp$. Entonces $\langle P_{V_j} f, g\rangle =$

$$= \langle P_{V_j} f, P_{V_j} g + P_{V_j^\perp} g\rangle = \langle P_{V_j} f, P_{V_j} g\rangle + \langle P_{V_j} f, P_{V_j^\perp} g\rangle$$

$$= \langle P_{V_j} f, P_{V_j} g\rangle + \langle P_{V_j^\perp} f, P_{V_j} g\rangle = \langle P_{V_j} f + P_{V_j^\perp} f, P_{V_j} g\rangle$$

$$= \langle f, P_{V_j} g\rangle.$$

$f \in L^2(\mathbb{R})$, $P_{V_j} f \in V_j \overset{\text{Prop 4.3.3}}{\Longrightarrow} P_{V_j} f \overset{(L^2(\mathbb{R}))}{=} \sum_{k\in\mathbb{Z}} \langle P_{V_j} f, \varphi_{j,k}\rangle \varphi_{j,k}$

$$= \sum_{k\in\mathbb{Z}} \langle f, \underbrace{P_{V_j}\varphi_{j,k}}_{\in V_j}\rangle \varphi_{j,k} = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{j,k}\rangle \varphi_{j,k} \quad (\text{en } L^2(\mathbb{R}))$$

Como $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$, $\lim_{j\to\infty} \|P_{V_j} f - f\|_2 = 0$. Por tanto, $P_{V_j} f$ es una aproximación a f con los coef. $\{\langle f, \varphi_{j,k}\rangle\}_{k\in\mathbb{Z}}$