Cap 4. Ondículas ortonormales (Wavelets)

4.1. INTRODUCCIÓN Y DEFINICIONES

En el ejercicio 1.5.4 se ha probado que $\{f_{n,k} : n, k \in \mathbb{Z}\}$ es base o.n. de $L^2(\mathbb{R})$, donde $f_{n,k}(x) = e^{2\pi i n x} \chi_{[0,1]}(x-k)$.

Def 4.1.1. $f : \mathbb{R} \to \mathbb{C}$. El operador <u>traslación</u> por $y_0 \in \mathbb{R}$ es $(T_{y_0} f)(x) = f(x - y_0)$. El operador <u>modulación</u> por $\omega_0 \in \mathbb{R}$, es $(M_{\omega_0} f)(x) = e^{2\pi i \omega_0 x} f(x)$.

Ejercicio 4.1.1 Prueba que los operaciones T_{y_0} y M_{ω_0} son lineales y unitarias en $L^2(\mathbb{R})$ y si $f \in L^1(\mathbb{R})$ se tiene

$$\mathcal{F}(T_{y_0} f)(\omega) = M_{-y_0} \mathcal{F} f(\omega)$$

$$\mathcal{F}(M_{\omega_0} f)(\omega) = T_{\omega_0} \mathcal{F} f(\omega)$$

donde $\mathcal{F}$ denota la transformada de Fourier.

S/ $$\|T_{y_0} f\|_2^2 = \int_{-\infty}^{\infty} |T_{y_0} f(x)|^2 dx = \int_{-\infty}^{\infty} |f(x-y_0)|^2 dx \underset{=}{\overset{x-y_0=y}{}}$$

$$= \int_{-\infty}^{\infty} |f(y)|^2 dy = \|f\|_2^2 .$$

$$\|M_{\omega_0} f\|_2^2 = \int_{-\infty}^{\infty} |M_{\omega_0} f(x)|^2 dx = \int_{-\infty}^{\infty} |e^{2\pi i \omega_0 x} f(x)|^2 dx$$

$$= \int_{-\infty}^{\infty} |f(x)|^2 dx = \|f\|_2^2 .$$

• $$\mathcal{F}(T_{y_0} f)(\omega) = \int_{-\infty}^{\infty} T_{y_0} f(x)\, e^{-2\pi i \omega x} dx =$$

$$= \int_{-\infty}^{\infty} f(x-y_0)\, e^{-2\pi i \omega x} dx \overset{x-y_0=y}{=\!=} \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i \omega (y+y_0)} dy$$

$$= e^{-2\pi i \omega y_0} \underbrace{\int_{-\infty}^{\infty} f(y)\, e^{-2\pi i \omega y} dy} = e^{-2\pi i \omega y_0} \mathcal{F}f(\omega) = M_{-y_0} \mathcal{F}f(\omega)$$

- $$\mathcal{F}(M_{\omega_0} f)(\omega) = \int_{-\infty}^{\infty} (M_{\omega_0} f)(x)\, e^{-2\pi i \omega x} dx =$$

$$= \int_{-\infty}^{\infty} e^{2\pi i \omega_0 x} f(x)\, e^{-2\pi i \omega x} dx = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i (\omega - \omega_0) x} dx$$

$$= \mathcal{F}f(\omega - \omega_0) = T_{\omega_0} \mathcal{F}f(\omega). \quad \blacksquare$$

Def. 4.1.2. $f: \mathbb{R} \to \mathbb{C}$. El operador <u>dilatación</u> por $a>0$ es $D_a f(x) = a^{1/2} f(ax)$.

<u>Ejercicio 4.1.2.</u> Probar que D_a es un operador lineal y unitario en $L^2(\mathbb{R})$ y si $f \in L^1(\mathbb{R})$

$$\mathcal{F}(D_a f)(\omega) = D_{1/a} \mathcal{F}f(\omega)$$

S/

$$\| D_a f \|_2^2 = \int_{-\infty}^{\infty} |D_a f(x)|^2 dx = \int_{-\infty}^{\infty} | a^{1/2} f(ax)|^2 dx$$

$$= \int_{-\infty}^{\infty} a\, |f(ax)|^2 dx \overset{ax=y}{=\!=} \int_{-\infty}^{\infty} a\, |f(y)|^2 \frac{dy}{a} = \int_{-\infty}^{\infty} |f(y)|^2 dy = \|f\|_2^2$$

- $$\mathcal{F}(D_a f)(\omega) = \int_{-\infty}^{\infty} D_a f(x)\, e^{-2\pi i \omega x} dx = \int_{-\infty}^{\infty} a^{1/2} f(ax)\, e^{-2\pi i \omega x} dx$$

$$\overset{ax=y}{=\!=} a^{1/2} \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i \omega \frac{y}{a}} \frac{dy}{a} = \frac{1}{a^{1/2}} \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i \frac{\omega}{a} \cdot y} dy$$

$$= a^{-1/2} \mathcal{F}f(a^{-1}\omega) = D_{a^{-1}} \mathcal{F}f(\omega) = D_{1/a} \mathcal{F}f(\omega). \quad \blacksquare$$

ψ = psi ③

Def 4.1.3 Una función $\psi \in L^2(\mathbb{R})$ es una ondícula ortonormal (orthonormal wavelet) para $L^2(\mathbb{R})$ si el conjunto

$$\{\psi_{j,k}(x) = D_{2^j} T_k f(x) = 2^{j/2} T_k f(2^j x) = 2^{j/2} f(2^j x - k) : j,k \in \mathbb{Z}\}$$

es base o.n. de $L^2(\mathbb{R})$.

Ejercicio 4.1.3. Comprobar que $D_{2^j} T_k f \neq T_k D_{2^j} f$

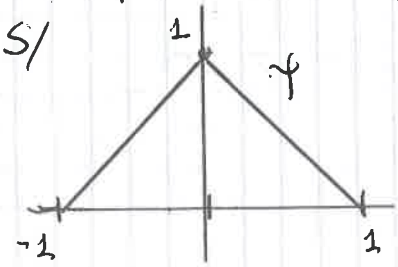
S/ $D_{2^j} T_k f(x) = 2^{j/2} f(2^j x - k)$, pero

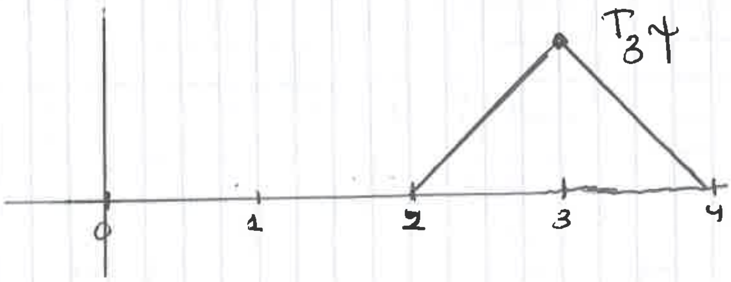
$$T_k D_{2^j} f(x) = D_{2^j} f(x-k) = 2^{j/2} f(2^j(x-k))$$
$$= 2^{j/2} f(2^j x - 2^j k)$$

Ejercicio 4.1.4 Sea $\psi(x) = \begin{cases} x+1 & \text{si } -1 \le x \le 0 \\ 1-x & \text{si } 0 < x \le 1 \\ 0 & \text{resto} \end{cases}$

Dibujar ψ, $T_3\psi$, $D_4\psi$, $D_4 T_3 \psi$ y $T_3 D_4 \psi$

S/





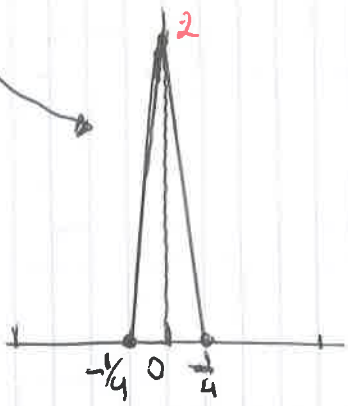
$D_4\psi(x) = 2\psi(4x)$

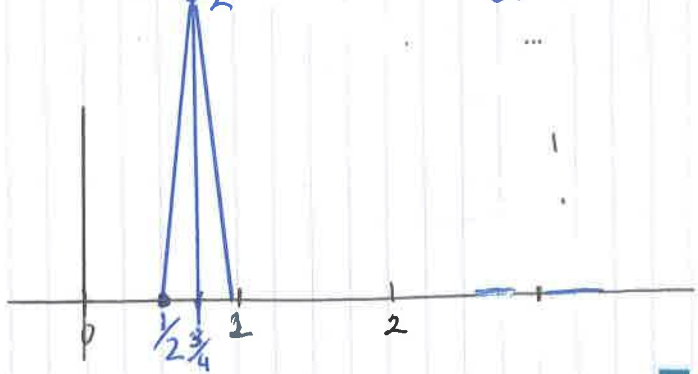
$-1 \le 4x \le 1 \Leftrightarrow -\frac{1}{4} \le x \le \frac{1}{4}$

$D_4 T_3 \psi(x) = 2\psi(4x-3)$

$-1 \le 4x-3 \le 1$; $2 \le 4x \le 4$

$\frac{1}{2} \le x \le 1$



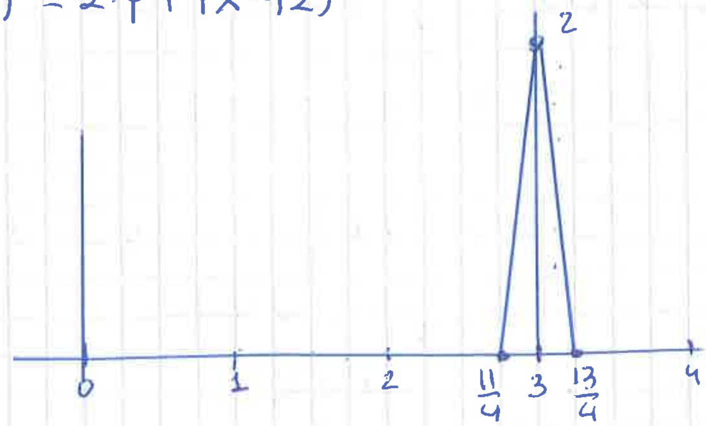


$$T_3 D_4 \psi(x) = 2\psi(4(x-3)) = 2\psi(4x-12)$$

$$-1 \leq 4x-12 \leq 1$$

$$11 \leq 4x \leq 13$$

$$\frac{11}{4} \leq x \leq \frac{13}{4}$$



Ejercicio 4.1.5 Probar

(a) $\|\psi_{j,k}\|_2 = \|\psi\|_2$

(b) $\mathcal{F}(\psi_{j,k})(\omega) = 2^{-j/2} e^{-2\pi i k \frac{\omega}{2^j}} \mathcal{F}\psi(\omega/2^j)$

S/ (a) $\|\psi_{j,k}\|_2 = \|D_{2^j} T_k \psi\|_2 \overset{Ej\ 4.1.2}{=} \|T_k\psi\|_2 \overset{Ej\ 4.1.1}{=} \|\psi\|_2$

(b) $\mathcal{F}(\psi_{j,k})(\omega) = \mathcal{F}(D_{2^j} T_k \psi)(\omega) = D_{2^{-j}} \mathcal{F}(T_k\psi)(\omega) =$

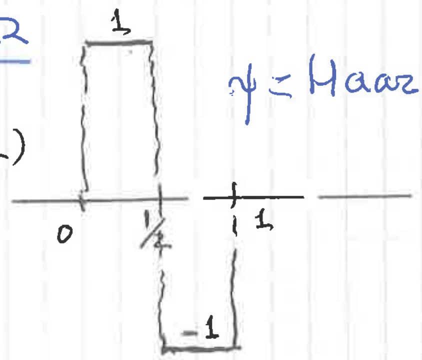
$$= D_{2^{-j}} M_{-k} \mathcal{F}\psi(\omega) = 2^{-j/2}(M_{-k}\mathcal{F}\psi)(2^{-j}\omega) =$$

$$= 2^{-j/2} e^{-2\pi i k \frac{\omega}{2^j}} \mathcal{F}\psi(\omega/2^j)$$

4.2. Las Ondículas de Haar y de Shannon

4.2.1. La Ondícula de Haar

$$\psi(x) = \begin{cases} 1 & \text{si } 0 \leq x \leq 1/2 \\ -1 & \text{si } 1/2 < x \leq 1 \\ 0 & \text{resto} \end{cases} \qquad (4.2.1)$$



ψ = Haar

$$\|\psi\|_2^2 = \int_0^{1/2} |1|^2 dx + \int_{1/2}^1 |-1|^2 dx$$

$$= \int_0^1 1\, dx = 1$$

Por el ejercicio 4.1.5, $\|\psi_{j,k}\|_2 = 1 \quad \forall j,k \in \mathbb{Z}$.

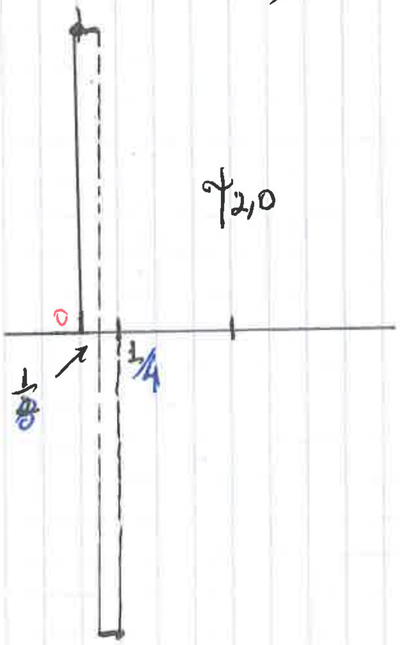
Observar: $\int_{-\infty}^{\infty} \psi(x)\,dx = \int_0^{1/2} 1\,dx + \int_{1/2}^{1} (-1)\,dx = \frac{1}{2} - \frac{1}{2} = 0$

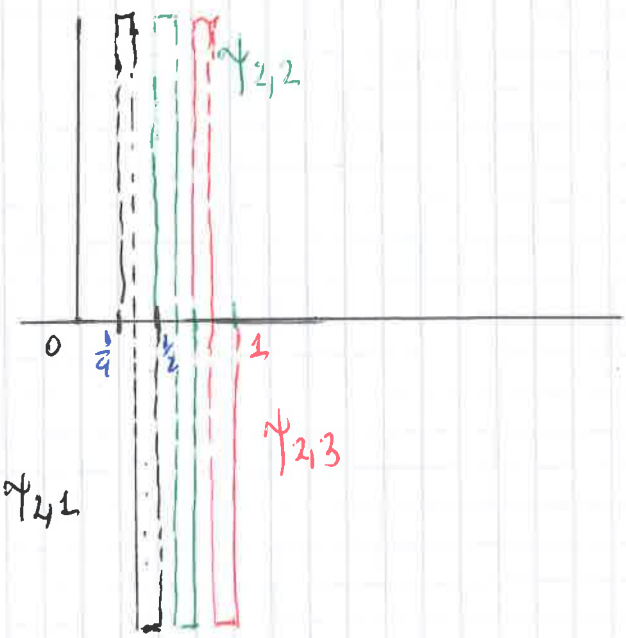
Ejercicio 4.2.1. Para la ondícula de Haar, dibujar

$\psi_{2,0}$; $\psi_{2,1}$; $\psi_{2,2}$ y $\psi_{2,3}$

S/ $\psi_{2,0}(x) = D_{2^2} T_0 \psi(x) = 2\psi(4x)$; $\psi_{2,1}(x) = D_{2^2} T_1(x) = 2\psi(4x-1)$

$0 \leq 4x \leq 1$; $0 \leq x \leq 1/4$ $\qquad$ $0 \leq 4x-1 \leq 1$; $\frac{1}{4} \leq x \leq \frac{1}{2}$





Proposición 4.2.1. Con ψ dada por (4.2.1), el conjunto $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ es un sistema o.n. en $L^2(\mathbb{R})$.