15/07/2021

Cap. 3. Bases Ortonormales para Señales e Imagenes

3.1. NUEVAS BASES EN $L^2([0,1])$.

Sabemos

a) $\{e^{2\pi i k x}\}_{k=-\infty}^{\infty}$ es b.o.n. de $L^2([0,1])$ (Cor. 1.6.5)

b) $\{\frac{1}{\sqrt{T}} e^{\frac{2\pi i k x}{T}}\}_{k=-\infty}^{\infty}$ es b.o.n de $L^2([a,b])$, $b-a=T$ (Ej. 1.6.6)

c) $\{f_{n,k}(x) = e^{2\pi i n x}\chi_{[k,k+1)}(x)\}$ es b.o.n. de $L^2(\mathbb{R})$ (Ej. 1.5.4)

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1. Aunque la f sea real, las operaciones tienen n.^{os} complejos

$(a+bi)+(c+di) = (a+c)+(b+d)i$

$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$ (6 operaciones)

2. El decaimiento de los coeficientes es lento. Por ejemplo, si $f\in C^2([0,1))$ sus coeficientes en la base a) son

$$\hat{f}(k) = \int_0^1 f(x)\, e^{-2\pi i k x}\, dx \qquad \left\{\begin{array}{l} u=f(x), \quad du=f'(x)dx \\ dv = e^{-2\pi i k x}dx, \quad v=\dfrac{e^{-2\pi i k x}}{-2\pi i k}\end{array}\right.$$

$$= \left[f(x)\frac{e^{-2\pi i k x}}{-2\pi i k}\right]_0^1 + \int_0^1 f'(x)\frac{e^{-2\pi i k x}}{2\pi i k}dx \qquad \left\{\begin{array}{l} u=f'(x), \quad du=f''(x)dx \\ dv = e^{-2\pi i k x}dx \\ v=\dfrac{e^{-2\pi i k x}}{-2\pi i k}\end{array}\right.$$

$$= \frac{f(1)-f(0)}{-2\pi i k} + \frac{1}{2\pi i k}\left[f'(x)\frac{e^{-2\pi i k x}}{-2\pi i k}\right]_0^1$$

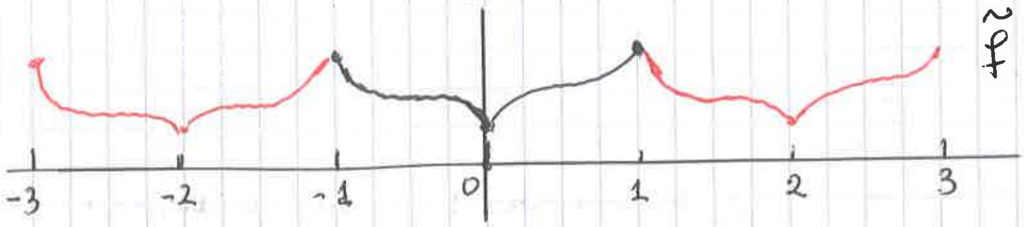
$$+ \int_0^1 f''(x)\frac{e^{-2\pi i k x}}{-4\pi^2 k^2}dx = \frac{f(1)-f(0)}{-2\pi i k} + \frac{1}{4\pi^2 k^2}[f'(1)-f'(0)]$$

$$- \frac{1}{4\pi^2 k^2}\int_0^1 f''(x)\, e^{-2\pi i k x}dx. \quad \text{Si } f(1)\neq f(0),$$

$\hat{f}(k) \approx \frac{A_f}{k}$ $(k\uparrow\infty)$; la serie $\sum_{k=1}^{\infty}\frac{1}{k}$ no converge.

3.1.1. Base de cosenos - I en $L^2([0,1])$

$f: [0,1] \to \mathbb{R}$; extender f a $\tilde{f}$ que sea 2-periódica, coincida con f en $[0,1]$ y sea simétrica respecto a $x=0$.



$$\tilde{f}(x) = \begin{cases} f(x) & \text{si } x \in [0,1] \\ f(-x) & \text{si } x \in [-1,0] \end{cases}$$

Una base o.n. de $L^2([-1,1])$ es $\left\{\frac{1}{\sqrt{2}} e^{\frac{2\pi i k x}{2}}\right\}_{k=-\infty}^{\infty} = \left\{\frac{1}{\sqrt{2}} e^{\pi i k x}\right\}_{k=-\infty}^{\infty}$

$k=0$, $\frac{1}{\sqrt{2}}$; $e^{\pi i k x} = \cos(\pi k x) + i \operatorname{sen} \pi k x)$, $k \neq 0$. Como

$$\tilde{f}(x) = \sum_{k=-\infty}^{\infty} c_k e^{\pi i k x} \quad (\text{en } L^2([-1,1]))$$

se puede escribir (3.1.1)

$$\tilde{f}(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\pi k x) + \sum_{k=1}^{\infty} b_k \operatorname{sen}(\pi k x) \quad (\text{en } L^2[-1,1])$$

Ejercicio 3.1.1. Comprobar que $\left\{\frac{1}{\sqrt{2}}, \cos(\pi k x), \operatorname{sen}(\pi k x)\right\}_{k=1}^{\infty}$ es b.o.n. de $L^2([-1,1])$.

S/ (3.1.1) prueba la completitud.

$$\int_{-1}^{1} \frac{1}{2} dx = 1 \ ; \quad \int_{-1}^{1} \cos^2(\pi k x) dx = \int_{-1}^{1} \frac{1}{2}\left[1 + \cos 2\pi k x\right] dx = 1$$

y $\int_{-1}^{1} \operatorname{sen}^2(\pi k x) dx = 1$

$k, l \neq 0$

$$\int_{-1}^{1} \cos(\pi k x) \operatorname{sen}(\pi l x) dx = \int_{-1}^{1} \frac{e^{\pi i k x} + e^{-\pi i k x}}{2} \cdot \frac{e^{\pi i l x} - e^{-\pi i l x}}{2i} dx$$

$$= \frac{1}{4i} \int_{-1}^{1} \left[e^{\pi i (k+l) x} - e^{\pi i (k-l) x} + e^{\pi i (l-k) x} - e^{-\pi i (k+l) x}\right] dx = 0$$

Como $b_k = \int_{-1}^{1} \tilde{f}(x)\, \text{sen}(\pi k x)\, dx = 0$ $\left\{ \begin{array}{l} \tilde{f} \text{ es par} \\ \text{sen}(\pi k x) \text{ es impar} \\ \tilde{f}(x)\,\text{sen}(\pi k x) \text{ es impar} \end{array} \right.$

porque $\tilde{f}(x)\,\text{sen}(\pi k x)$ es impar

y $[-1,1]$ es un intervalo simétrico

respecto a cero.

Como $\tilde{f}(x) = f(x)$ cuando $x \in [0,1]$, (3.1.1) $\Rightarrow$

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\pi k x) \quad (\text{en } L^2([0,1])) \quad (3.1.2)$$

Proposición 3.1.2. (Base de cosenos - I)

El conjunto $\{1, \sqrt{2}\cos(\pi k x)\}_{k=1}^{\infty}$ es b.o.n. de $L^2([0,1])$.

D/ (3.1.2) prueba que es completo.

$$\int_0^1 1\, dx = 1, \quad \int_0^1 2[\cos(\pi k x)]^2 dx = 2\int_0^1 \frac{1+\cos(2\pi k x)}{2} dx = 1$$

$k \neq l$

$$2\int_0^1 \cos(\pi k x)\cos(\pi l x)\, dx = \dots\dots = 0$$

Ejercicio 3.1.2 Si $f \in C^2(\mathbb{R})$ y $\lambda_k := \int_0^1 f(x)\cos(\pi k x)\, dx$,

se tiene que $|\lambda_k| \leq \dfrac{A_f}{k^2}$ $\left(\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \text{ es convergente} \right)$

S/

$$\lambda_k = \int_0^1 f(x)\cos(\pi k x)\, dx \qquad \left\{ \begin{array}{l} u = f(x), \quad du = f'(x)\,dx \\ dv = \cos(\pi k x)\,dx, \quad v = \dfrac{\text{sen}(\pi k x)}{\pi k} \end{array} \right.$$

$$= \cancel{\left[f(x) \frac{\text{sen}(\pi k x)}{\pi k} \right]_0^1}^{\,0} - \int_0^1 f'(x) \frac{\text{sen}(\pi k x)}{\pi k}\, dx \qquad \left\{ \begin{array}{l} u = f'(x), \quad du = f''(x)\,dx \\ dv = \text{sen}(\pi k x)\,dx, \quad v = \dfrac{-\cos(\pi k x)}{\pi k} \end{array} \right.$$

$$= +\left[f'(x) \frac{\cos(\pi k x)}{(\pi k)^2} \right]_0^1 + \int_0^1 f''(x) \frac{\cos(\pi k x)}{(\pi k)^2}\, dx$$

$$= \frac{-f'(0) + (-1)^k f'(1)}{(\pi k)^2} + \int_0^1 f''(x) \frac{\cos(\pi k x)}{(\pi k)^2}\, dx$$

$$|\lambda_k| \leq \frac{|f'(1)| + |f'(0)|}{\pi^2 k^2} + \|f''\|_{\infty} \frac{1}{\pi^2 k^2} = \frac{A_f}{k^2}$$

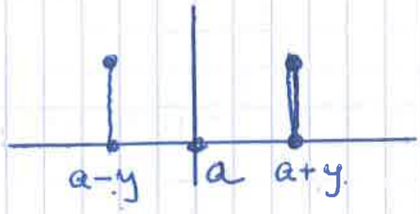
3.1.2. Base de cosenos - IV en $L^2([0,1])$

Ejercicio 3.1.3 a) Probar que si $f: \mathbb{R} \to \mathbb{R}$ es simétrica respecto a $x=a$, se tiene $f(x) = f(2a-x)$

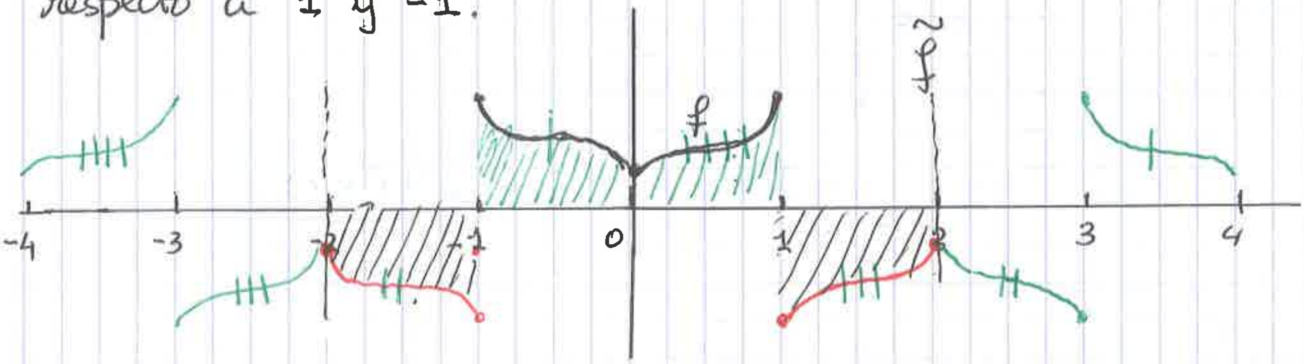
b) Probar que si $f: \mathbb{R} \to \mathbb{R}$ es antisimétrica respecto a $x=a$, se tiene $f(x) = -f(2a-x)$

S/ (a) Para todo $y \in \mathbb{R}$, $f(a+y) = f(a-y)$. Sea $x = a-y$ $(y = a-x)$

$$f(2a-x) = f(x)$$



$f \in L^2([0,1])$. Extender f a $\tilde{f}$ definida en $\mathbb{R}$, periódica de periodo 4, simétrica respecto al origen y antisimétrica respecto a 1 y -1.



$$\tilde{f}(x) = \begin{cases} f(x) & \text{si } x \in [0,1] \\ f(-x) & \text{si } x \in [-1,0] \\ -f(2-x) & \text{si } x \in (1,2] \\ -f(2+x) & \text{si } x \in [-2,-1) \end{cases} \quad \text{4-periódica}$$

Una base de $L^2([-2,2])$ es: $\left\{ \frac{1}{2} e^{\frac{2\pi i k x}{4}} \right\}_{k=-\infty}^{\infty} = \left\{ \frac{1}{2} e^{\frac{\pi i k x}{2}} \right\}_{k=-\infty}^{\infty}$

$$\tilde{f}(x) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{\pi i k x}{2}} \quad (\text{en } L^2([-2,2]))$$

$$\tilde{f}(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{\pi k x}{2} + \sum_{k=1}^{\infty} b_k \operatorname{sen}\frac{\pi k x}{2} \quad (\text{en } L^2([-2,2])) \qquad (3.1.3)$$

$$a_0 = \frac{1}{4}\int_{-2}^{2} \tilde{f}(x)\,dx = 0$$

$$b_k = \frac{1}{2}\int_{-2}^{2} \tilde{f}(x) \operatorname{sen}\frac{\pi k x}{2}\,dx = 0 \quad \text{pq } \tilde{f}(x) \operatorname{sen}\frac{\pi k x}{2} \text{ es impar}$$

$$a_{2k} = \frac{1}{2}\int_{-2}^{2} \tilde{f}(x) \cos\frac{\pi(2kx)}{2}\,dx = \frac{1}{2}\int_{-2}^{2} \tilde{f}(x) \cos(\pi k x)\,dx$$

$$= \frac{1}{2}\Big[-\int_{-2}^{-1} f(2+x)\cos(\pi k x)\,dx + \int_{-1}^{0} f(-x)\cos(\pi k x)\,dx + \int_{0}^{1} f(x)\cos(\pi k x)\,dx$$

$2+x=y$ $\qquad -x=y$

$$-\int_{1}^{2} f(2-x)\cos(\pi k x)\,dx\Big] = \frac{1}{2}\Big[-\cancel{\int_{0}^{1} f(y)\cos(\pi k y)\,dy} + \cancel{\int_{0}^{1} f(y)\cos(\pi k y)\,dy}$$

$2-x=y$

$$+\cancel{\int_{0}^{1} f(y)\cos(\pi k y)\,dy} - \cancel{\int_{0}^{1} f(y)\cos(\pi k y)\,dy}\Big] = 0$$

Por tanto,

$$\tilde{f}(x) = \sum_{k=0}^{\infty} a_{2k+1} \cos\frac{\pi(2k+1)x}{2} \quad (\text{en } L^2([-2,2]))$$

Como $f(x)$ coincide con $\tilde{f}(x)$ en $[0,1]$

$$f(x) = \sum_{k=0}^{\infty} a_{2k+1} \cos\frac{\pi(2k+1)x}{2} \quad (\text{en } L^2([0,1])) \tag{3.1.4}$$

Proposición 3.1.3 (Base de cosenos - IV)

El conjunto $\left\{\sqrt{2}\cos\frac{\pi(2k+1)x}{2}\right\}_{k=1}^{\infty}$ es b.o.n. de $L^2([0,1])$

D/ Es completo por (3.1.4).

$$2\int_0^1 \left[\cos\frac{\pi(2k+1)x}{2}\right]^2 dx = 2\int_0^1 \frac{1+\cos\pi(2k+1)x}{2}\,dx = 1$$

$k \neq l$

$$2\int_0^1 \cos\frac{\pi(2k+1)x}{2}\cos\frac{\pi(2l+1)x}{2}\,dx = 0$$

3.2. NUEVAS BASES ORTONORMALES DISCRETAS

$S_N(\{0,..,N-1\}) = \{ (f(n))_{n=0}^{N-1} \} \approx \mathbb{R}^N$ - Señales discretas reales

Para $0 \leq k \leq N-1$, $e_k(n) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i k n}{N}}$, $n=0,..,N-1$. Sabemos que $\{e_k\}_{k=0}^{N-1}$ es b.o.n. de $(S_N, <,>)$ donde

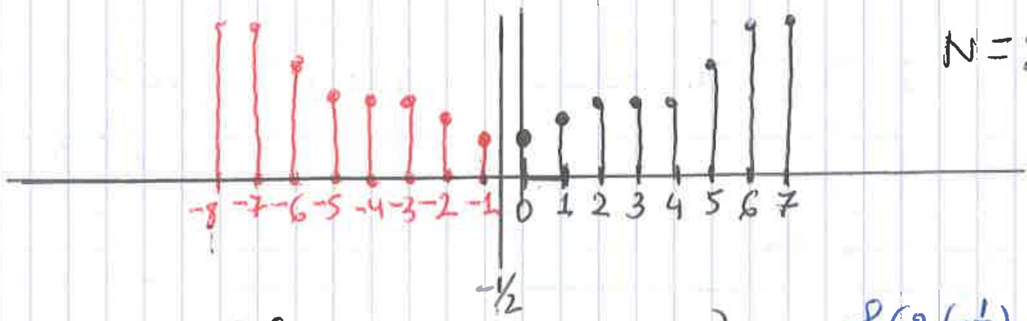
$$<f,g> = \sum_{n=0}^{N-1} f(n)\overline{g(n)} \qquad \text{(Prop. 2.5.1)}$$

- $e_k = (e_k(0), e_k(1), ..., e_k(n), ..., e_k(N-1))$

$$= \frac{1}{\sqrt{N}}\left(1, e^{\frac{2\pi i k}{N}}, ..., e^{\frac{2\pi i k n}{N}}, ..., e^{\frac{2\pi i k(N-1)}{N}}\right)$$

3.2.1 Base discreta de cosenos I (DC-I)

$f=(f_n)_{n=0}^{N-1} \in S_N$. Extender a una señal $\tilde{f}$ de tamaño $2N$ que sea simétrica respecto a $-\frac{1}{2}$



$N=8$

$$\tilde{f}(n) = \begin{cases} f(n) & \text{si } 0 \leq n \leq N-1 \\ f(-1-n) & \text{si } -N \leq n \leq -1 \end{cases} \qquad f(2(-\tfrac{1}{2})-n) = f(-1-n)$$

es una señal de $S_{2N}(\{-N, ..., N-1\})$

Ejercicio 3.2.1. Para $-N \leq k \leq N-1$, sea $u_k^{(2N)}(n) = \frac{1}{\sqrt{2N}} e^{\frac{k\pi i}{N}(n+\frac{1}{2})}$, $-N \leq n \leq N-1$. Probar que $\{u_k^{(2N)}\}_{k=-N}^{N-1}$ es base o.n de $S_{2N}(\{-N, .., N-1\})$

$$S/ \ <u_k^{(2N)}, u_k^{(2N)}> = \frac{1}{2N} \sum_{n=-N}^{N-1} 1 = \frac{1}{2N} 2N = 1$$

$$k \neq \ell \quad \langle u_k^{(2N)}, u_\ell^{(2N)} \rangle = \frac{1}{2N} \sum_{n=-N}^{N-1} e^{\frac{(k-\ell)\pi i}{N}(n+\frac{1}{2})} \quad \text{(prog geométrica)}$$

$$= \frac{1}{2N} \frac{e^{\frac{(k-\ell)\pi i}{N}(N-\frac{1}{2})} e^{\frac{(k-\ell)\pi i}{N}} - e^{\frac{(k-\ell)\pi i}{N}(-N+\frac{1}{2})}}{e^{\frac{(k-\ell)\pi i}{N}} - 1} \qquad e^{(k-\ell)\pi i} = (-1)^{k-\ell}$$

$$= \frac{1}{2N} \frac{(-1)^{k-\ell} e^{\frac{(k-\ell)\pi i}{2N}} - (-1)^{-(k-\ell)} e^{\frac{(k-\ell)\pi i}{2N}}}{e^{\frac{(k-\ell)\pi i}{N}} - 1} = 0$$

Es base porque tiene $2N$ elementos, que es la dimensión de S_{2N}

$e^{i\theta} = \cos\theta + i \operatorname{sen}\theta$. Los vectores

$$\left(C_k^{(2N)}(n)\right)_{n=-N}^{N-1} := \left(\cos \frac{\pi k}{N}(n+\tfrac{1}{2})\right)_{n=-N}^{N-1}, \quad k=-N, \dots, N-1$$

y

$$\left(S_k^{(2N)}(n)\right)_{n=-N}^{N-1} := \left(\operatorname{sen} \frac{\pi k}{N}(n+\tfrac{1}{2})\right)_{n=-N}^{N-1}, \quad k=-N, \dots, N-1$$

Son sistema de generadores de S_{2N}, pero no puede ser base porque hay $4N$ elementos en un espacio de dimensión $2N$

Teorema 3.2.1 (Base discreta de cosenos - I, DC-I)
El conjunto $\left\{ \lambda_k \sqrt{\frac{2}{N}} \left(C_k^{(2N)}(n) \right)_{n=0}^{N-1} \right\}_{k=0}^{N-1}$ con $\lambda_k = \begin{cases} 1/\sqrt{2} & \text{si } k=0 \\ 1 & \text{si } 1 \le k \le N-1 \end{cases}$
es base o.n. de S_N.

<u>Ejercicio 3.2.2</u> Probar que $\displaystyle\sum_{n=0}^{N-1} \cos\left(\frac{\pi k}{N}(n+\tfrac{1}{2})\right) = 0$
para $k = 1, 2, \dots, 2N-1$

$$\text{S/} \sum_{n=0}^{N-1} \cos\left(\frac{\pi k}{N}(n+\tfrac{1}{2})\right) = \frac{1}{2} \sum_{n=0}^{N-1} \left[e^{\frac{\pi k i}{N}(n+\frac{1}{2})} + e^{-\frac{\pi k i}{N}(n+\frac{1}{2})} \right]$$

$$= \frac{1}{2} e^{\frac{\pi k i}{2N}} \left[\sum_{n=0}^{N-1} e^{\frac{\pi k i n}{N}} + e^{-\frac{\pi k i}{N}\underbrace{(n+1)}_{m}} \right]$$

$$= \frac{1}{2} e^{\frac{\pi k i}{2N}} \left[\sum_{n=0}^{N-1} e^{\frac{\pi k i n}{N}} + \sum_{m=1}^{N} e^{-\frac{\pi k i}{N} m} \right]$$

$$= \frac{1}{2} e^{\frac{\pi k i}{2N}} \left[\sum_{n=0}^{N-1} e^{\frac{\pi k i n}{N}} + \sum_{n=-N}^{-1} e^{\frac{\pi k i n}{N}} \right]$$

$$= \frac{1}{2} e^{\frac{\pi k i}{2N}} \left[\sum_{n=-N}^{N-1} e^{\frac{\pi k i n}{N}} \right]$$ progre. geométrica

$k \neq 0$
$$= \frac{1}{2} e^{\frac{\pi k i}{2N}} \frac{e^{\frac{\pi k i}{N}(N-1)} \cdot e^{\frac{\pi k i}{N}} - e^{-\frac{\pi k i N}{N}}}{e^{\frac{\pi k i}{N}} - 1}$$

$$= \frac{1}{2} e^{\frac{\pi k i}{2N}} \frac{e^{\pi k i} - e^{-\pi k i}}{e^{\frac{\pi k i}{N}} - 1} = \frac{1}{2} e^{\frac{\pi k i}{2N}} \frac{(-1)^k - (-1)^{-k}}{e^{\frac{\pi k i}{N}} - 1} = 0$$

Dem (del Teorema 3.2.1). Para $k = 1, 2, \dots, N-1$

$$\langle C_0^{(2N)}, C_k^{(2N)} \rangle = \sum_{n=0}^{N-1} \lambda_0 \sqrt{\tfrac{2}{N}}\, \lambda_k \sqrt{\tfrac{2}{N}} \cos \frac{k\pi}{N}\left(n+\tfrac{1}{2}\right)$$

$$= \frac{1}{\sqrt{2}} \frac{2}{N} \sum_{n=0}^{N-1} \cos \frac{k\pi}{N}\left(n+\tfrac{1}{2}\right) = 0$$

$k \neq l$, $k, l \neq 0$

$$\langle C_k^{(2N)}, C_l^{(2N)} \rangle = \sum_{n=0}^{N-1} \lambda_k \sqrt{\tfrac{2}{N}} \cos \frac{k\pi}{N}\left(n+\tfrac{1}{2}\right) \lambda_l \sqrt{\tfrac{2}{N}} \cos \frac{l\pi}{N}\left(n+\tfrac{1}{2}\right)$$

$$= \frac{2}{N} \sum_{n=0}^{N-1} \frac{1}{2} \left[\cos \frac{(k+l)\pi}{N}\left(n+\tfrac{1}{2}\right) + \cos \frac{(k-l)\pi}{N}\left(n+\tfrac{1}{2}\right) \right]$$

$$= \frac{1}{N} \left[\sum_{n=0}^{N-1} \cos \frac{(k+l)\pi}{N}\left(n+\tfrac{1}{2}\right) + \sum_{n=0}^{N-1} \cos \frac{(k-l)\pi}{N}\left(n+\tfrac{1}{2}\right) \right] = 0$$

por el ejercicio 3.2.2 ya que $0 < k+l \leq 2N-1$ y $0 \leq k-l \leq N$

$$\|C_0\|^2 = \langle C_0, C_0 \rangle = \sum_{n=0}^{N-1} \left(\lambda_0 \sqrt{\tfrac{2}{N}} \cdot 1\right)^2 = \sum_{n=0}^{N-1} \frac{1}{N} = 1$$

$k \neq 0$

$$\|C_k\|^2 = \langle C_k, C_k \rangle = \sum_{n=0}^{N-1} \left(\lambda_k \sqrt{\tfrac{2}{N}} \cos \frac{k\pi}{N}\left(n+\tfrac{1}{2}\right)\right)^2 =$$

$$= \frac{2}{N} \sum_{n=0}^{N-1} \frac{1 + \cos \frac{2k\pi}{N}\left(n+\frac{1}{2}\right)}{2} = \frac{1}{N} \sum_{n=1}^{N-1} 1 + \cos \frac{2k\pi}{N}\left(n+\tfrac{1}{2}\right)$$

$= 1 + 0$ por el ejercicio 3.2.2 ya que $0 < 2k \leq 2N-2$ ∎

Con respecto a la base $\overbrace{\left\{ \lambda_k \sqrt{\frac{2}{N}} \left(\cos \frac{k\pi}{N}(n+\frac{1}{2})\right)_{n=0}^{N-1} \right\}_{k=0}^{N-1}}^{o.n}$

de S_N, si $f \in S_N$ se puede escribir

$$f(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} a_k \lambda_k \cos\left[\frac{k\pi}{N}(n+\frac{1}{2})\right], \quad n=0,-,N-1$$

(3.2.1)

donde

$$a_k = \langle f, \lambda_k \sqrt{\frac{2}{N}} e_k^{(2N)} \rangle = \lambda_k \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} f(n) \cos\left[\frac{\pi k}{N}(n+\frac{1}{2})\right]$$

$k=0,-,N-1$

que se llaman la transformada discreta de coseno - I (DCT-I)

Def 3.2.2. Si $f \in S_N$, su <u>transformada discreta de coseno - I</u> (DCT-I) es $(\hat{f}_I(k))_{k=0}^{N-1}$ donde para $0 \le k \le N-1$

$$\hat{f}_I(k) = \lambda_k \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} f(n) \cos\left[\frac{\pi k}{N}(n+\frac{1}{2})\right]$$

(3.2.2)

con $\lambda_k = \left\{ \begin{array}{ll} 1/\sqrt{2} & \text{si } k=0 \\ 1 & \text{si } 1 \le k \le N-1 \end{array} \right\}$.

De (3.2.1) se deduce que si $n=0,1,-,N-1$

$$f(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} \hat{f}_I(k) \lambda_k \cos\left[\frac{k\pi}{N}(n+\frac{1}{2})\right] \quad (3.2.3)$$

que es la inversa de la CDT-I (IDCT-I)

Ejercicio 3.2.3. Escribir DCT-I en S_2 $(N=2)$ en forma matrices

S/ $\hat{f}_I(0) = \frac{1}{\sqrt{2}}[f(0) + f(1)]$; $\hat{f}_I(1) = [f(0)\cos\frac{\pi}{4} + f(1)\cos\frac{3\pi}{4}]$

$$\begin{pmatrix} \hat{f}_I(0) \\ \hat{f}_I(1) \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{C} \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}; \quad C.C^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow C^{-1} = C^T$$

$$\begin{pmatrix} f(0) \\ f(1) \end{pmatrix} = C^T \begin{pmatrix} \hat{f}_I(0) \\ \hat{f}_I(1) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \hat{f}_I(0) \\ \hat{f}_I(1) \end{pmatrix}.$$

Ejercicio 3.2.4 Escribir DCT-I y su inversa en S_3 $(N=3)$ en forma matricial

S/ $\hat{f}_I(0) = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}\left[f(0)+f(1)+f(2)\right]$

$$\hat{f}_I(1) = \sqrt{\frac{2}{3}}\left[f(0)\cos\frac{\pi}{6} + f(1)\cos\frac{3\pi}{6} + f(2)\cos\frac{5\pi}{6}\right]$$

$$= \sqrt{\frac{2}{3}}\left[f(0)\frac{\sqrt{3}}{2} + f(1)\cdot 0 + f(2)\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$\hat{f}_I(2) = \sqrt{\frac{2}{3}}\left[f(0)\cos\frac{\pi}{3} + f(1)\cos\pi + f(2)\cos\frac{5\pi}{3}\right]$$

$$= \sqrt{\frac{2}{3}}\left[f(0)\frac{1}{2} + f(1)(-1) + f(2)\left(-\frac{1}{2}\right)\right]$$

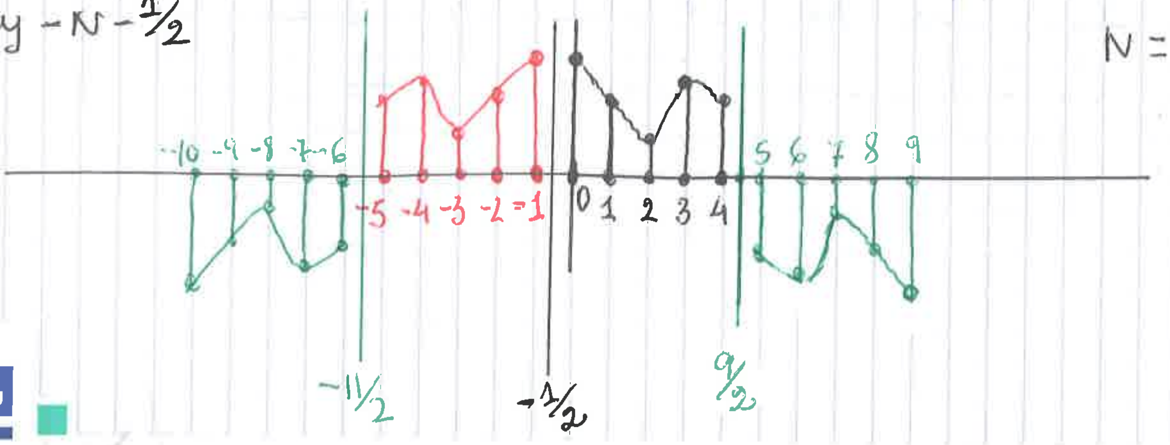
$$\begin{pmatrix}\hat{f}_I(0)\\ \hat{f}_I(1)\\ \hat{f}_I(2)\end{pmatrix} = \underbrace{\sqrt{\frac{2}{3}}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2}\\ \frac{1}{2} & -1 & -\frac{1}{2}\end{pmatrix}}_{C}\begin{pmatrix}f(0)\\ f(1)\\ f(2)\end{pmatrix} \quad ; \quad C\cdot C^T = I, \quad C^{-1} = C^T$$

$$\begin{pmatrix}f(0)\\ f(1)\\ f(2)\end{pmatrix} = \sqrt{\frac{2}{3}}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{\sqrt{2}} & 0 & -1\\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\hat{f}_I(0)\\ \hat{f}_I(1)\\ \hat{f}_I(2)\end{pmatrix} \quad \text{(Inversa)}$$

3.2.2. Base discreta de cosenos - IV

$f \in S_N$; extender f a una señal $\tilde{f}$ de tamaño $4N$, que sea simétrica respecto a $-\frac{1}{2}$ y antisimétrica respecto a $N-\frac{1}{2}$ y $-N-\frac{1}{2}$

$N=5$



Ejercicio 3.2.5 Para $-2N \leq k \leq 2N-1$, sea

$$u_k^{(4N)}(n) = \frac{1}{\sqrt{4N}} e^{\frac{k\pi i}{2N}(n+\frac{1}{2})}, \quad n=-2N,\dots,2N-1.$$

Probar que $\{u_k^{(4N)}\}_{k=-2N}^{2N-1}$ es una base o.n. de $S_{4N}(\{-2N,\dots,2N-1\})$

S/ $$\langle u_k^{(4N)}, u_k^{(4N)} \rangle = \frac{1}{4N} \sum_{n=-2N}^{2N-1} e^{\frac{k\pi i}{N}(n+\frac{1}{2})} e^{-\frac{k\pi i}{N}(n+\frac{1}{2})} = 1$$

$k \neq \ell$

$$\langle u_k^{(4N)}, u_\ell^{(4N)} \rangle = \frac{1}{4N} \sum_{n=-2N}^{2N-1} e^{(k-\ell)\frac{\pi i}{N}(n+\frac{1}{2})} = \frac{1}{4N} \sum_{n=-2N}^{2N-1} e^{(k-\ell)\frac{\pi i}{N}n} \cdot e^{(k-\ell)\frac{\pi i}{2N}}$$

$$= \frac{1}{4N} e^{(k-\ell)\frac{\pi i}{2N}} \sum_{n=-2N}^{2N-1} e^{((k-\ell)\frac{\pi i}{N}n} = \frac{1}{4N} e^{(k-\ell)\frac{\pi i}{2N}} \frac{e^{(k-\ell)\frac{\pi i}{N}2N} - e^{-(k-\ell)\frac{\pi i}{N}2N}}{e^{(k-\ell)\frac{\pi i}{N}} - 1}$$

$$= \frac{1}{4N} e^{(k-\ell)\frac{\pi i}{2N}} \frac{e^{(k-\ell)2\pi i} - e^{-(k-\ell)2\pi i}}{e^{(k-\ell)\frac{\pi i}{N}} - 1} = 0.$$

Es base porque tiene $4N$ elementos, que es la dim. de S_{4N}

$$e^{\frac{k\pi i}{2N}(n+\frac{1}{2})} = \cos\left(\frac{k\pi}{2N}(n+\tfrac{1}{2})\right) + i \operatorname{sen}\left(\frac{k\pi}{2N}(n+\tfrac{1}{2})\right)$$

el conjunto

$$\left(C_k^{(4N)}(n)\right)_{n=-2N}^{2N-1} = \left(\cos \frac{k\pi}{2N}(n+\tfrac{1}{2})\right)_{n=-2N}^{2N-1}, \quad k=-2N,\dots,2N-1$$

y

$$\left(S_k^{(4N)}(n)\right)_{n=-2N}^{2N-1} = \left(\operatorname{sen} \frac{k\pi}{2N}(n+\tfrac{1}{2})\right)_{n=-2N}^{2N-1}, \quad k=-2N,\dots,2N-1$$

genera S_{4N}. Pero ~~tiene~~ tiene $8N$ elementos; sobran $4N$ de estos