① 14/07/2021

Ejercicio 2.5.3 Sea $f = (f(0), f(1), f(2), f(3)) = (1, 2, 3, -1)$
una señal de tamaño 4

(a) Calcular DFT de f directamente

(b) Calcular DFT de f usando FFT

S/ (a) $\hat{f} = (\hat{f}(0), \hat{f}(1), \hat{f}(2), \hat{f}(3)) = \left(\frac{5}{2}, \frac{-2-3i}{2}, \frac{3}{2}, \frac{-2+3i}{2}\right)$

(b) $_0f(n) = \frac{1}{\sqrt{2}}\left[f(n) + f(n+\frac{N}{2})\right]$; $_1f(n) = \frac{1}{\sqrt{2}} e^{\frac{-2\pi i n}{N}}\left[f(n) - f(n+\frac{N}{2})\right]$

$N=4$

$_0f(0) = \frac{1}{\sqrt{2}}[f(0)+f(2)] = \frac{1}{\sqrt{2}}[1+3] = \frac{4}{\sqrt{2}}$

$_0f(1) = \frac{1}{\sqrt{2}}[f(1)+f(3)] = \frac{1}{\sqrt{2}}[2-1] = \frac{1}{\sqrt{2}}$

$_1f(0) = \frac{1}{\sqrt{2}}[f(0)-f(2)] = \frac{1}{\sqrt{2}}[1-3] = -\frac{2}{\sqrt{2}}$

$_1f(1) = \frac{1}{\sqrt{2}} e^{\frac{-\pi i}{2}}[f(1)-f(3)] = \frac{1}{\sqrt{2}}(-i)[2-(-1)] = \frac{-3i}{\sqrt{2}}$

$_0f = \left(\frac{4}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ | $_1f = \left(-\frac{2}{\sqrt{2}}, \frac{-3i}{\sqrt{2}}\right)$

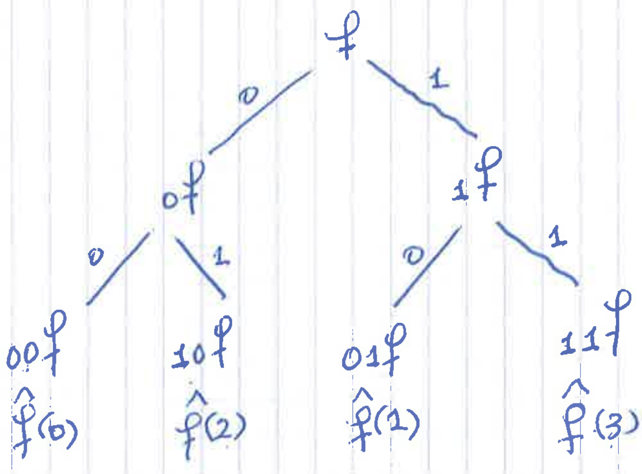
$N=2$

$_0(_0f)(0) = \frac{1}{\sqrt{2}}[_0f(0) + {_0f}(1)]$
$= \frac{1}{\sqrt{2}}\left[\frac{4}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = \frac{5}{2}$

$_1(_0f)(0) = \frac{1}{\sqrt{2}}\left[\frac{4}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] = \frac{3}{2}$

$_0(_1f)(0) = \frac{1}{\sqrt{2}}\left[-\frac{2}{\sqrt{2}} - \frac{3i}{\sqrt{2}}\right] = \frac{-2-3i}{2}$

$_1(_1f)(0) = \frac{1}{\sqrt{2}}\left[-\frac{2}{\sqrt{2}} + \frac{3i}{\sqrt{2}}\right] = \frac{-2+3i}{2}$



2.6. MUESTREO DE IMÁGENES

b

0 a

- Una imagen es una función
$$f : [0,a]\times[0,b] \longrightarrow \mathbb{R}$$
que se puede extender a $\mathbb{R}\times\mathbb{R}$ con valores cero fuera de $[0,a]\times[0,b]$

- Los valores $f(x,y)$ deben representar el color de la imagen en el punto (x,y) (más sobre el color en el cap 3)

- Dados $\{t_n\}\subset[0,a]$ y $\{s_m\}\subset[0,b]$ las muestras de $f(x,y)$ en estos puntos son
$$\{f(t_n,s_m)\}_{n,m}$$

Necesitamos

- Transformada de Fourier en $\mathbb{R}^2$
- Series y coeficientes de Fourier en $[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}] = [-\frac{1}{2},\frac{1}{2}]^2$

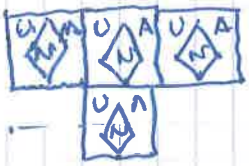
<u>Def 2.6.1</u> $f:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$; la transformada de Fourier de $f\in L^1(\mathbb{R}^2)$ es
$$\mathcal{F}f(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)\, e^{-2\pi i(x_1\omega_1+x_2\omega_2)}\, dx_1\, dx_2$$

$\mathcal{F}f$ en $\mathbb{R}\times\mathbb{R}$ tiene propiedades similares al caso $\mathbb{R}$. En particular se puede extender por densidad a $\mathcal{F}: L^2(\mathbb{R}\times\mathbb{R})\to L^2(\mathbb{R}\times\mathbb{R})$ y que cumple Plancherel: $\|\mathcal{F}f\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)}$ y la de Parseval: $\langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2(\mathbb{R}^2)} = \langle f, g\rangle_{L^2(\mathbb{R}^2)}$ $\forall f,g\in L^2(\mathbb{R}^2)$

Su inversa
$$\mathcal{F}^{-1}g(x_1,x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\omega_1,\omega_2)\, e^{2\pi i(x_1\omega_1+x_2\omega_2)}\, d\omega_1\, d\omega_2$$

La fórmula de inversión:
$$f(x_1,x_2) = \mathcal{F}^{-1}(\mathcal{F}f)(x_1,x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}f(\omega_1,\omega_2)\, e^{2\pi i(x_1\omega_1+x_2\omega_2)}\, d\omega_1\, d\omega_2 .$$

Def 2.6.2. $f:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ es periódica, de periodo 1, si
$f(x_1+1,\, x_2+1)=f(x_1,x_2) \quad \forall\, x_1, x_2\in\mathbb{R}$



$L^1_p([-\frac{1}{2},\frac{1}{2}]^2)=\{f:\mathbb{R}\times\mathbb{R}\to\mathbb{C}:$
f es 1-per y $\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}|f(x_1,x_2)|\,dx_1dx_2<\infty$

Igual para $L^2_p([-\frac{1}{2},\frac{1}{2}])$.

Proposición 2.6.3. Sea $e_{k_1,k_2}(x_1,x_2)=e^{2\pi i(x_1k_1+x_2k_2)}=$
$=e^{2\pi i x_1k_1}e^{2\pi i x_2 k_2}=e_{k_1}(x_1)\,e_{k_2}(x_2)$, $(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}$, $(x_1,x_2)\in\mathbb{R}\times\mathbb{R}$.
El conjunto $\{e_{k_1,k_2}\}_{k_1,k_2=-\infty}^{\infty}$ es una base o.n. de $L^2([-\frac{1}{2},\frac{1}{2}]^2)$

P/ El producto interno en $L^2([-\frac{1}{2},\frac{1}{2}]^2)$ es

$$\langle f,g\rangle=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} f(x_1,x_2)\,\overline{g(x_1,x_2)}\,dx_1dx_2.$$

$$\langle e_{k_1,k_2},e_{m_1,m_2}\rangle=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{2\pi i(x_1k_1+x_2k_2)}e^{-2\pi i(x_1m_1+x_2m_2)}\,dx_1dx_2$$

$$=\left(\int_{-1/2}^{1/2}e^{2\pi i x_1(k_1-m_1)}dx_1\right)\left(\int_{-1/2}^{1/2}e^{2\pi i x_2(k_2-m_2)}dx_2\right)=\delta_{k_1,m_1}\delta_{k_2,m_2}$$

$(=1$ si $(k_1,k_2)=(m_1,m_2))$

$D_2=\{f_1(x_1)f_2(x_2): f_1\in L^2_p([-1/2,1/2]),\ f_2\in L^2_p([-1/2,1/2])\}$ es denso en $L^2_p([-1/2,1/2]^2)$. Como $\{e_{k_1}(x_1)\}_{k_1=-\infty}^{\infty}$ es completo en $L^2_p([-1/2,1/2])$ y $\{e_{k_2}(x_2)\}_{k_2=-\infty}^{\infty}$ es completo en $L^2_p([-1/2,1/2])$, $\{e_{k_1,k_2}(x_1,x_2)=e_{k_1}(x_1)\,e_{k_2}(x_2)\}_{k_1,k_2=-\infty}^{\infty}$ es completo en $L^2_p([-1/2,1/2]^2)$. ∎

Def. 2.6.4 $f \in L^1_p([-\frac{1}{2},\frac{1}{2}]^2)$. Para $(k_1,k_2) \in \mathbb{Z}\times\mathbb{Z}$ se define el coeficiente de Fourier de f como

$$\hat{f}(k_1,k_2) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} f(x_1,x_2)\, e^{-2\pi i (x_1k_1+x_2k_2)}\, dx_1 dx_2$$

La serie de Fourier de f es

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \hat{f}(k_1,k_2)\, e^{2\pi i (x_1k_1+x_2k_2)}$$

Si $f \in L^2_p([-\frac{1}{2};\frac{1}{2}]^2)$, con respecto a la base $\{e_{k_1,k_2}\}_{k_1,k_2=-\infty}^{\infty}$ de la Prop. 2.6.3 se puede escribir

$$f(x_1,x_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \lambda_{k_1,k_2}\, e_{k_1,k_2}(x_1,x_2) \quad (\text{en } L^2([-\tfrac{1}{2},\tfrac{1}{2}]^2))$$

con

$$\lambda_{k_1,k_2} = \langle f, e_{k_1,k_2}\rangle_2 = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} f(x_1,x_2)\, e^{-2\pi i (k_1x_1+k_2x_2)}\, dx_1 dx_2 = \hat{f}(k_1,k_2)$$

Es decir,

$$f(x_1,x_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \hat{f}(k_1,k_2)\, e^{2\pi i (x_1k_1+x_2k_2)} \quad (\text{en } L^2([-\tfrac{1}{2},\tfrac{1}{2}]^2)). \tag{2.6.1}$$

Teorema 2.6.5. (Muestreo en $\mathbb{R}\times\mathbb{R}$; Whittaker-Shannon)
$f \in L^2(\mathbb{R}\times\mathbb{R})$ con sop $\mathcal{F}f \subset [-\frac{1}{2},\frac{1}{2}]^2$ (soporte compacto).
Entonces

$$f(x_1,x_2) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n,m)\, \frac{\operatorname{sen}\pi(x_1-n)}{\pi(x_1-n)} \cdot \frac{\operatorname{sen}\pi(x_2-m)}{\pi(x_2-m)}$$

convergencia en $L^2(\mathbb{R}\times\mathbb{R})$ y uniformemente en $\mathbb{R}\times\mathbb{R}$.

D/ Definir $F_p(\omega_1,\omega_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \mathcal{F}f(\omega_1+k_1, \omega_2+k_2)$,

que es 1-per en $\mathbb{R}\times\mathbb{R}$ y bien definida pq $\operatorname{sop} \mathcal{F}f \subset [-\frac{1}{2},\frac{1}{2}]^2$

Se tiene
$$F_p \cdot \chi_{[-\frac{1}{2},\frac{1}{2}]^2} = \mathcal{F}f$$

y $F_p \in L^2_p([-\frac{1}{2},\frac{1}{2}]^2)$: $\|F_p\|^2_{L^2} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |F_p(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2$

$$= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |\mathcal{F}f(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{F}f(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2 =$$

$$= \|\mathcal{F}f\|^2_{L^2(\mathbb{R}^2)} \overset{\text{Plancherel}}{=} \|f\|^2_{L^2(\mathbb{R}^2)} < \infty$$

Escribir la serie de Fourier de $F_p(\omega_1,\omega_2)$ usando (2.6.1)

$$F_p(\omega_1,\omega_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \hat{F}_p(k_1,k_2)\, e_{k_1,k_2}(\omega_1,\omega_2) \quad (\text{en } L^2([-\tfrac{1}{2},\tfrac{1}{2}]^2))$$

donde

$$\hat{F}_p(k_1,k_2) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \underbrace{F_p(\omega_1,\omega_2)}_{\mathcal{F}f(\omega_1,\omega_2)} e^{-2\pi i k_1\omega_1} e^{-2\pi i k_2 \omega_2} d\omega_1 d\omega_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}f(\omega_1,\omega_2)\, e^{-2\pi i(k_1\omega_1 + k_2\omega_2)} d\omega_1 d\omega_2 \overset{\text{Inversa}}{=}$$

$$= \mathcal{F}^{-1}(\mathcal{F}f)(-k_1,-k_2) = f(-k_1,-k_2)$$

Entonces

$$\hat{F}_p(\omega_1,\omega_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} f(-k_1,-k_2)\, e^{2\pi i x_1 k_1} e^{2\pi i x_2 k_2}$$

y

$$\mathcal{F}f(\omega_1,\omega_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} f(-k_1,-k_2)\, e^{2\pi i x_1 k_1} e^{2\pi i x_2 k_2} \chi_{[-\frac{1}{2},\frac{1}{2}]^2}(\omega_1,\omega_2)$$

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} f(k_1,k_2)\left(e^{-2\pi i x_1 k_1}\chi_{[-\frac{1}{2},\frac{1}{2}]}(\omega_1)\right)\left(e^{-2\pi i x_2 k_2}\chi_{[-\frac{1}{2},\frac{1}{2}]}(\omega_2)\right)$$

Haciendo $\mathcal{F}^{-1}$:

$$f(x_1,x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} f(k_1,k_2)\, \mathcal{F}^{-1}\left(e^{-2\pi i k_1 \cdot} \chi_{[-1/2,1/2]}\right)(x_1) \cdot \mathcal{F}^{-1}\left(e^{-2\pi i k_2 \cdot} \chi_{[-1/2,1/2]}\right)(x_2)$$

$$\underbrace{=}_{\text{Ejercicio 2.4.1}} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} f(k_1,k_2) \frac{\operatorname{sen} \pi(x_1-k_1)}{\pi(x_1-k_1)} \frac{\operatorname{sen} \pi(x_2-k_2)}{\pi(x_2-k_2)}.$$

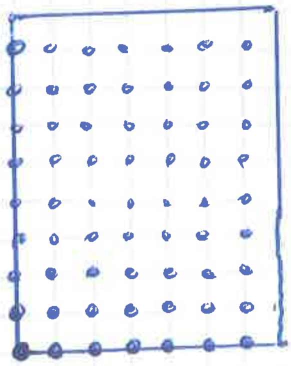
Ejercicio 2.6.1 (Para entregar el 19/07, segunda fecha)

Probar que si $f \in L^2(\mathbb{R}\times\mathbb{R})$ y $\operatorname{sop} \mathcal{F}f \subset [-\frac{T_1}{2}, \frac{T_1}{2}]\times[-\frac{T_2}{2}, \frac{T_2}{2}]$, $(T_1, T_2 > 0)$ entonces

$$f(x_1,x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} f\left(\frac{k_1}{T_1}, \frac{k_2}{T_2}\right) \frac{\operatorname{sen} \pi(T_1x_1-k_1)}{\pi(T_1x_1-k_1)} \frac{\operatorname{sen} \pi(T_2x_2-k_2)}{\pi(T_2x_2-k_2)}$$

con convergencia en $L^2(\mathbb{R}\times\mathbb{R})$ y uniforme en $\mathbb{R}\times\mathbb{R}$.

Indicación: Definir $g(x_1,x_2) = \frac{1}{T_1}\frac{1}{T_2} f\left(\frac{x_1}{T_1}, \frac{x_2}{T_2}\right)$. Probar que $g \in L^2(\mathbb{R}\times\mathbb{R})$ y $\operatorname{sop} \mathcal{F}g \subset [-1/2, 1/2]$. Usar el teorema de muestreo de Wittacker-Shannon (Teor 2.6.5) y hacer dos cambios de variable.



$S_{N,M} = \{ f : \{0,1,\dots,N-1\}\times\{0,1,\dots,M-1\} \to \mathbb{C} \}$

$\simeq \mathbb{C}^{N\times M}$ espacio vectorial de dim. finita $N\times M$ y es de Hilbert

$$\langle f, g\rangle = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)\,\overline{g(n,m)}$$

Proposición 2.6.6. $e_{k,\ell}(n,m) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i k n}{N}} \frac{1}{\sqrt{M}} e^{\frac{2\pi i \ell m}{M}} = e_k(n)\cdot e_\ell(m)$

$k, n = 0, \dots, N-1$, $\ell, m = 0, \dots, M-1$. El conjunto $\{e_{k,\ell}\}_{k=0}^{N-1},{}_{\ell=0}^{M-1}$ es base o.n. de $S_{N,M}$.

$f \in S_{N,M}$; con respecto a la base de la Prop 2.6.6,

$$f(n,m) = \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \langle f, e_{k,\ell} \rangle e_{k,\ell}(n,m) \qquad (2.6.2)$$

Def 2.6.7 Para $(k,\ell) \in \mathbb{Z} \times \mathbb{Z}$, ~~los~~ números

$$\hat{f}(k,\ell) = \langle f, e_{k,\ell} \rangle = \frac{1}{\sqrt{N}} \frac{1}{\sqrt{M}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f(n,m)\, e^{\frac{-2\pi i n k}{N}} e^{-\frac{2\pi i m \ell}{M}}$$

son la (2D)-DFT (Discrete Fourier Transform 2D)

La formula (2.6.2) da la inversa: (2D)-IDFT

- Operaciones para calcular (2D)-DFT de una imagen $f \in S_{N\times M}$ son $2N^2 \times M^2$
- Para reducir la complejidad del cálculo de la (2D)-DFT se puede hacer la DFT en cada variable usando FFT en cada variable.

$$\hat{f}(k,\ell) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \Bigg(\underbrace{\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n,m)\, e^{-\frac{2\pi i n k}{N}}}_{\substack{f(\hat{k},m) \\ \text{DFT en la var } 1^{a}}} \Bigg) e^{-\frac{2\pi i m \ell}{M}} = \underbrace{\frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f(\hat{k}, m)\, e^{-\frac{2\pi i m \ell}{M}}}_{\substack{f(\hat{k},\hat{\ell}) \\ \text{DFT en la var } 2^{a}}}$$

Con este algoritmo la complejidad de (2D)-DFT es

$$\frac{3}{2} NM (\log_2 N + \log_2 M)$$