Prop. 2.1.2. (Derivadas y transformada de Fourier)

(a) Si $x^k f(x) \in L^1(\mathbb{R})$ para $k = 0, 1, 2, \dots, n$ entonces $\mathcal{F}f \in C^n(\mathbb{R})$ y $\dfrac{d^k \mathcal{F}f}{d\omega^k}(\omega) = \mathcal{F}\left((-2\pi i x)^k f\right)(\omega)$ $k = 1, 2, \dots n$

(b) Si $f \in C^n(\mathbb{R}) \cap L^1(\mathbb{R})$ y $f^{(k)} \in L^1(\mathbb{R})$, $k = 1, 2, \dots, n$, entonces $\mathcal{F}\left(f^{(k)}\right)(\omega) = (2\pi i \omega)^k \mathcal{F}f(\omega)$

(c) Si $f \in L^1(\mathbb{R})$ y tiene soporte compacto, $\mathcal{F}f \in C^\infty(\mathbb{R})$

- f tiene soporte compacto si $\exists\, [a,b] \subset \mathbb{R}$ tal que $f(x) = 0$ si $x \notin [a,b]$

D/ (a)

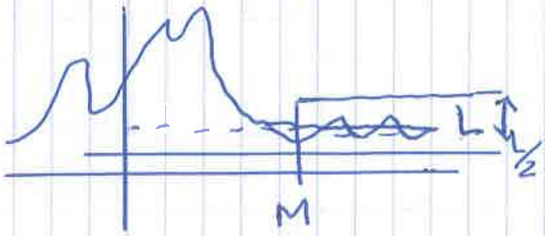
$$\frac{d^k \mathcal{F}f}{d\omega^k}(\omega) = \frac{d^k}{d\omega^k} \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\omega}\, dx = \int_{-\infty}^{\infty} f(x) \frac{d^k}{d\omega^k}\left(e^{-2\pi i x\omega}\right) dx$$

$$= \int_{-\infty}^{\infty} \underbrace{f(x)\, (-2\pi i x)^k}\, e^{-2\pi i x\omega}\, dx = \mathcal{F}\left((-2\pi i x)^k f\right)(\omega)$$

10-07-2021

NOTA: Si $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, $\lim_{|x| \to \infty} f(x) = 0$

Si $\lim_{x \to \infty} f(x) = L \neq 0$. Como f es continua, $\exists\, M > 0$ tal que $|f(x)| > \frac{L}{2}$ para todo $x > M$.



Entonces,

$$\int_{-\infty}^{\infty} |f(x)|\, dx \geq \int_M^{\infty} |f(x)|\, dx > \int_M^{\infty} \frac{L}{2}\, dx = \infty$$

que es una contradicción.

(b) $k=1$

$$\mathcal{F}(f')(\omega) = \int_{-\infty}^{\infty} f'(x)\, e^{-2\pi i x\omega}\, dx$$

$$\underline{\underline{\text{Partes}}} \left[f(x)\, e^{-2\pi i x\omega} \right]_{-\infty}^{\infty} \qquad \left\{ \begin{array}{l} u = e^{-2\pi i x\omega} \\ dv = f'(x)\,dx \\ du = (-2\pi i\omega)\, e^{-2\pi i x\omega}\, dx \\ v = f(x) \end{array} \right\}$$

$$- \int_{-\infty}^{\infty} f(x)\,(-2\pi i\omega)\, e^{-2\pi i x\omega}\, dx = \int_{-\infty}^{\infty} (2\pi i\omega)\, f(x)\, e^{-2\pi i x\omega}\, dx$$

$$= (2\pi i\omega)\, \mathcal{F}f(\omega)$$

$$\lim_{x\to\infty} \left| f(x)\, e^{-2\pi i x\omega} \right| = \lim_{x\to\infty} |f(x)| \overset{\text{NOTA}}{=} 0$$

Si $1<k\leq n$ usamos inducción:

$$\left\{ \begin{array}{l} u = e^{-2\pi i x\omega} \\ dv = \frac{d^k f}{dx^k}\, dx \\ du = (-2\pi i\omega)\, e^{-2\pi i x\omega}\, dx \\ v = \frac{d^{k-1} f}{dx^{k-1}}(x) \end{array} \right\}$$

$$\mathcal{F}\left(\frac{d^k f}{dx^k}\right)(\omega) = \int_{-\infty}^{\infty} \frac{d^k f}{\partial x^k}(x)\, e^{-2\pi i x\omega}\, dx$$

$$= \left[\frac{d^{k-1} f}{dx^{k-1}}(x)\, e^{-2\pi i x\omega} \right]_{-\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty} \frac{d^{k-1} f}{dx^{k-1}}(x)\, (-2\pi i\omega)\, e^{-2\pi i x\omega}\, dx = (2\pi i\omega)\, \mathcal{F}\left(\frac{d^{k-1} f}{dx^{k-1}}\right)(\omega)$$

$$\underline{\underline{\text{inducción}}} \quad (2\pi i\omega)(2\pi i\omega)^{k-1}\, \mathcal{F}(f)(\omega) = (2\pi i\omega)^k\, \mathcal{F}f(\omega).$$

(c) $f \in L^1(\mathbb{R})$ y con soporte compacto tenemos que: existe $M>0$ tal que $f(x)=0$ si $|x|>M$. Como

$$\int_{\mathbb{R}} |x|^k |f(x)|\, dx = \int_{-M}^{M} |x|^k |f(x)|\, dx \leq M^k \|f\|_1 < \infty$$

$x^k f(x) \in L^1(\mathbb{R})$ $\forall\, k=1,2,3,\ldots$. Por la parte (a), $\mathcal{F}f \in C^\infty(\mathbb{R})$

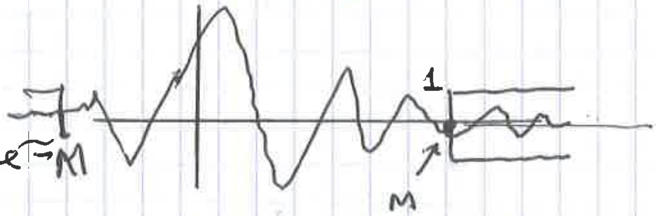
Lema 2.1.3. Sea $f \in C^2(\mathbb{R})$ tal que $f, f', f'' \in L^1(\mathbb{R})$. Entonces $\mathcal{F}f \in L^1(\mathbb{R})$

D/ Por la Prop 2.1.2 (b), $\mathcal{F}(f'')(\omega) = (2\pi i \omega)^2 \mathcal{F}f(\omega)$

Por el lema de Riemann-Lebesgue, $\lim_{|\omega|\to\infty} \mathcal{F}(f'') = 0$

(lo probamos después)

$$\lim_{|\omega|\to\infty} \omega^2 \mathcal{F}f(\omega) = 0.$$



Con $\varepsilon = 1$, existe $M > 0$ tal que

$|\omega|^2 |\mathcal{F}f(\omega)| \leq 1$ para todo $|\omega| \geq M$. Como $\mathcal{F}f$ es continua, está acotada en $[-M, M]$, e.d. $\exists R > 0$ t.q. $|\mathcal{F}f(\omega)| \leq R$ para todo $\omega \in [-M, M]$. Entonces

$$\int_{-\infty}^{\infty} |\mathcal{F}f(\omega)|\, d\omega = \int_{-M}^{M} |\mathcal{F}f(\omega)|\, d\omega + \int_{|\omega|>M} |\mathcal{F}f(\omega)|\, d\omega \leq$$

$$\leq \int_{-M}^{M} R\, d\omega + \int_{|\omega|>M} \frac{1}{\omega^2}\, d\omega = R(2M) + 2\int_{M}^{\infty} \frac{1}{\omega^2}\, d\omega$$

$$= 2RM + 2\frac{1}{M} < \infty \Rightarrow \mathcal{F}f \in L^1(\mathbb{R}) \quad \blacksquare$$

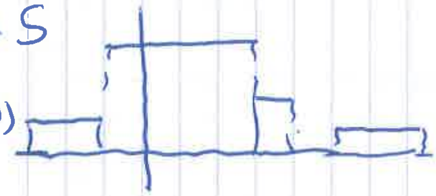
Lema 2.1.4 (Riemann-Lebesgue). Si $f \in L^1(\mathbb{R})$,

$$\lim_{|\omega|\to\infty} \mathcal{F}f(\omega) = 0$$

D/ Por el ejercicio 2.1.3, $\lim_{|\omega|\to\infty} \mathcal{F}\chi_{[a,b]}(\omega) = 0$. Sea

$$S = \left\{ f = \sum_{i=1}^{n} \alpha_i \chi_{[a_i, b_i]} : \alpha_i \in \mathbb{C},\ a_i, b_i \in \mathbb{R},\ n \in \mathbb{N} \right\}$$

el conjunto de funciones simples. Si $f \in S$



$$\lim_{|\omega|\to\infty} \mathcal{F}f(\omega) = \lim_{|\omega|\to\infty} \mathcal{F}\left(\sum_{i=1}^{n} \alpha_i \chi_{[a_i, b_i]} \right)(\omega)$$

$$= \lim_{|\omega|\to\infty} \sum_{i=1}^{n} \alpha_i \mathcal{F}(\chi_{[a_i, b_i]})(\omega) = 0.$$

③

El conjunto S es denso en $L^1(\mathbb{R})$. Dada $f \in L^1(\mathbb{R})$ existe $\{f_n\}_{n=1}^{\infty} \subset S$ tal que $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. Entonces

$$|\mathcal{F}f(\omega)| = |\mathcal{F}f(\omega) - \mathcal{F}f_n(\omega) + \mathcal{F}f_n(\omega)| \underset{\mathcal{F}\text{ lineal}}{\overset{DT}{\leq}} |\mathcal{F}(f-f_n)(\omega)| + |\mathcal{F}f_n(\omega)|$$

$$= \left|\int_{-\infty}^{\infty} (f(x) - f_n(x))\, e^{-2\pi i x\omega}\, dx\right| + |\mathcal{F}f_n(\omega)|$$

$$\leq \int_{-\infty}^{\infty} |f(x) - f_n(x)|\, dx + |\mathcal{F}f_n(\omega)| = \|f - f_n\|_1 + |\mathcal{F}f_n(\omega)|$$

Ahora, dado $\varepsilon > 0$, como $\lim_{n\to\infty} \|f_n - f\|_1 = 0$, existe $N \in \mathbb{N}$ t.q. $\|f_n - f\|_1 < \varepsilon/2$ cuando $n \geq N$. Fijado un $n \geq N$, como $f_n \in S$ y $\lim_{|\omega|\to\infty} \mathcal{F}(f_n)(\omega) = 0$, existe $M > 0$ tal que $|\mathcal{F}f_n(\omega)| < \varepsilon/2$ si $|\omega| > M$. Entonces

$$|\mathcal{F}f(\omega)| \leq \|f - f_n\|_1 + |\mathcal{F}f_n(\omega)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{si } |\omega| > M \quad \blacksquare$$

Def 2.1.5 Si $g \in L^1(\mathbb{R})$, definimos

$$\mathcal{F}^{-1}g(x) = \int_{-\infty}^{\infty} g(\omega)\, e^{2\pi i \omega x}\, d\omega = \mathcal{F}g(-x)$$

Teorema 2.1.6 (Inversa de $\mathcal{F}$ en $L^1(\mathbb{R})$)

Sea $f \in L^1(\mathbb{R})$ y $\mathcal{F}f \in L^1(\mathbb{R})$. Para casi todo punto $x \in \mathbb{R}$

$$\mathcal{F}^{-1}(\mathcal{F}f)(x) = \int_{-\infty}^{\infty} \mathcal{F}f(\omega)\, e^{2\pi i x\omega}\, d\omega = f(x).$$

Si $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ y $\mathcal{F}f \in L^1(\mathbb{R})$, la igualdad se cumple en todo punto $x \in \mathbb{R}$.

$$\mathcal{F}: \underset{x}{L^1(\mathbb{R})} \longrightarrow \underset{\omega}{C_0(\mathbb{R})}$$

$$\underset{x}{C_0(\mathbb{R})} \longleftarrow \underset{\omega}{L^1(\mathbb{R})} : \mathcal{F}^{-1}$$

2.2. CONVOLUCIÓN DE DOS FUNCIONES

Def 2.2.1. $f, g : \mathbb{R} \to \mathbb{C}$; la convolución de f y g, si existe, es

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \overset{x-y=u}{=\!=} \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du$$

NOTA: Si $f(x) = 1 = g(x)$, $(f*g)(x) = \int_{-\infty}^{\infty} 1\, dy = \infty$

Para que exista $f*g$ hay que poner condiciones sobre f y g.

Ejercicio 2.2.1 $f = g = \chi_{[-1/2, 1/2]}$. Calcule $f*g$

S/ $$f*g(x) = \int_{-\infty}^{\infty} \chi_{[-1/2,1/2]}(y)\, \chi_{[-1/2,1/2]}(x-y)\, dy = \int_{x-1/2}^{x+1/2} \chi_{[-1/2,1/2]}(y)\, dy$$

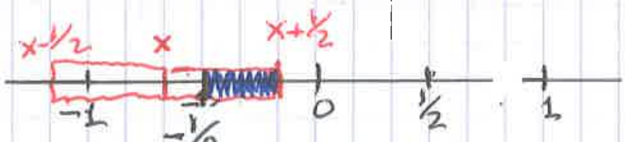
$$\chi_{[-1/2,1/2]}(x-y) = 1 \iff -\tfrac{1}{2} \le x-y \le \tfrac{1}{2} \iff x - \tfrac{1}{2} \le y \le x + \tfrac{1}{2}$$

$x + \frac{1}{2} < -1 + \frac{1}{2} = -\frac{1}{2}$

- Si $x < -1$, $[x-\frac{1}{2}, x+\frac{1}{2}] \cap [-\frac{1}{2}, \frac{1}{2}] = \emptyset$

Por tanto $f*g(x) = 0$

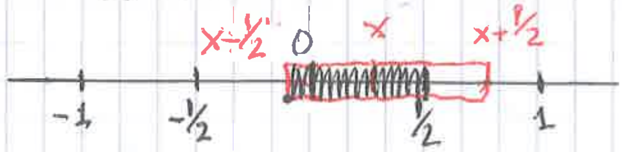
- Si $-1 \le x \le 0$,



$[x-\frac{1}{2}, x+\frac{1}{2}] \cap [-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{2}, x+\frac{1}{2}]$

Por tanto $f*g(x) = \int_{-1/2}^{x+1/2} 1\, dy = x + \frac{1}{2} - (-\frac{1}{2}) = x + 1$

- Si $0 < x \le 1$

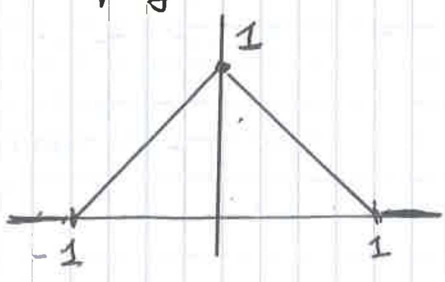


$[x-\frac{1}{2}, x+\frac{1}{2}] \cap [-\frac{1}{2}, \frac{1}{2}] = [x-\frac{1}{2}, \frac{1}{2}]$

Por tanto $f*g(x) = \int_{x-1/2}^{1/2} 1\, dy = 1 - x$

- Si $x > 1$, $[x-\frac{1}{2}, x+\frac{1}{2}] \cap [-\frac{1}{2}, \frac{1}{2}] = \emptyset \Rightarrow f*g(x) = 0$

$$\chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}(x) = \begin{cases} 0 & \text{si } x < -1 \\ x+1 & \text{si } -1 \le x \le 0 \\ 1-x & \text{si } 0 < x \le 1 \\ 0 & \text{si } x > 1 \end{cases}$$



Observe que $f*g$ es continua, aunque f y g son discontinuas.

Proposición 2.2.1 Si $f, g \in L^1(\mathbb{R})$, la convolución $f*g \in L^1(\mathbb{R})$ y $\|f*g\|_1 \leq \|f\|_1 \|g\|_1$

D/

$$\|f*g\|_1 = \int_{-\infty}^{\infty} |f*g(x)|\,dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy \right| dx$$

$$\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(y)|\,|g(x-y)|\,dy \right) dx \underbrace{=}_{x-y=t}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)|\,|g(t)|\,dy\,dt = \left(\int_{-\infty}^{\infty} |f(y)|\,dy \right) \left(\int_{-\infty}^{\infty} |g(t)|\,dt \right)$$

$$= \|f\|_1 \|g\|_1 .$$

Prop. 2.2.2. (Convolución y transformada de Fourier)

Si $f, g \in L^1(\mathbb{R})$, $\quad \mathcal{F}(f*g)(\omega) = \mathcal{F}f(\omega) \cdot \mathcal{F}g(\omega)$

D/ Prop 2.2.1 $\Rightarrow$ $f*g \in L^1(\mathbb{R})$. Se puede calcular $\mathcal{F}(f*g)$ usando la fórmula de la Def 2.1.1:

$$\mathcal{F}(f*g)(\omega) = \int_{-\infty}^{\infty} f*g(x)\, e^{-2\pi i x\omega}\,dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy \right) e^{-2\pi i x\omega}\,dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-y)\, \underbrace{e^{-2\pi i x\omega}\, e^{2\pi i y\omega}}_{e^{-2\pi i (x-y)\omega}}\,dx \right) f(y)\, e^{-2\pi i y\omega}\,dy$$

$$\underset{x-y=t}{=} \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} g(t)\, e^{-2\pi i t\omega}\,dt \right)}_{\mathcal{F}g(\omega)} f(y)\, e^{-2\pi i y\omega}\,dy = \mathcal{F}g(\omega) \cdot \mathcal{F}f(\omega)$$

NOTA: También para $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}(f*g) = \mathcal{F}^{-1}f \cdot \mathcal{F}^{-1}g$$

<u>Ejercicio 2.2.2</u>. Prueba $\displaystyle\int_{-\infty}^{\infty} \left(\frac{\operatorname{sen} t}{t}\right)^2 dt = \pi$

<u>Sugerencia</u>: Hallar la transformada de Fourier de $\chi_{[-\frac{1}{2},\frac{1}{2}]} * \chi_{[-\frac{1}{2},\frac{1}{2}]}$ y usar la fórmula de inversión de Fourier

S/ Sabemos que $\mathcal{F}(\chi_{[-\frac{1}{2},\frac{1}{2}]})(\omega) = \begin{cases} \dfrac{\operatorname{sen} \pi\omega}{\pi\omega} & \text{si } \omega \neq 0 \\ 1 & \text{si } \omega = 0 \end{cases}$.

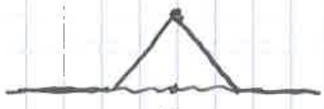
Entonces

$$\mathcal{F}(\chi_{[-\frac{1}{2},\frac{1}{2}]} * \chi_{[-\frac{1}{2},\frac{1}{2}]})(\omega) = \begin{cases} \left(\dfrac{\operatorname{sen} \pi\omega}{\pi\omega}\right)^2 & \text{si } \omega \neq 0 \\ 1 & \text{si } \omega = 0 \end{cases}$$

por la Prop 2.2.2. Observe que $\chi_{[-\frac{1}{2},\frac{1}{2}]} \in L^1(\mathbb{R})$.

Fórmula de inversión:

$$\int_{-\infty}^{\infty} \mathcal{F}(\chi_{[-\frac{1}{2},\frac{1}{2}]} * \chi_{[-\frac{1}{2},\frac{1}{2}]})(\omega)\, e^{2\pi i \omega x}\, d\omega = \chi_{[-\frac{1}{2},\frac{1}{2}]} * \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$$

$\chi_{[-\frac{1}{2},\frac{1}{2}]} * \chi_{[-\frac{1}{2},\frac{1}{2}]} =$ 

es continua

Para $x = 0$

$$\int_{-\infty}^{\infty} \left(\frac{\operatorname{sen} \pi\omega}{\pi\omega}\right)^2 d\omega = \chi_{[-\frac{1}{2},\frac{1}{2}]} * \chi_{[-\frac{1}{2},\frac{1}{2}]}(0) = 1$$

Entonces,

$$1 = \int_{-\infty}^{\infty} \left(\frac{\operatorname{sen} \pi\omega}{\pi\omega}\right)^2 d\omega \overset{\pi\omega = t}{=} \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\operatorname{sen} t}{t}\right)^2 dt \Rightarrow \int_{-\infty}^{\infty} \left(\frac{\operatorname{sen} t}{t}\right)^2 dt = \pi.$$

<u>Prop 2.2.3</u> $f \in L^1(\mathbb{R})$, $g \in L^2(\mathbb{R})$. Entonces $f * g \in L^2(\mathbb{R})$ y $\|f * g\|_2 \leq \|f\|_1 \|g\|_2$.

D/ $|f*g(x)|^2 = \left|\int_{-\infty}^{\infty} f(y)g(x-y)\,dy\right|^2 \leq \left(\int_{-\infty}^{\infty} |f(y)|\,|g(x-y)|\,dy\right)^2$

$= \left(\int_{-\infty}^{\infty} \underbrace{|f(y)|^{1/2}\,|g(x-y)|}\ \underbrace{|f(y)|^{1/2}}\,dy\right)^2$

$y \to |f(y)|^{1/2} \in L^2(\mathbb{R})$ p.q. $\int_{-\infty}^{\infty} (|f(y)|^{1/2})^2\,dy = \int_{-\infty}^{\infty} |f(y)|\,dy = \|f\|_1 < \infty$

$y \to |f(y)|^{1/2}|g(x-y)| \in L^2(\mathbb{R})$ p.q. $|f(y)|^{1/2} \in L^2(\mathbb{R})$ y

$g(x-y) \in L^2(\mathbb{R})$ y por Cauchy-Schwarz

$$\int_{-\infty}^{\infty} |f(y)|^{1/2}\,|g(x-y)| \leq \|f\|_1^{1/2}\|g\|_2 < \infty$$

Por Cauchy-Schwarz

$$|f*g(x)|^2 \leq \left(\int_{-\infty}^{\infty} |f(y)|\,|g(x-y)|^2\,dy\right)\left(\int_{-\infty}^{\infty} |f(y)|\,dy\right)$$

$$= \|f\|_1 \left(\int_{-\infty}^{\infty} |f(y)|\,|g(x-y)|^2\,dy\right)$$

Finalmente,

$$\|f*g\|_2^2 = \int_{-\infty}^{\infty} |f*g(x)|^2\,dx \leq \int_{-\infty}^{\infty} \|f\|_1 \left(\int_{-\infty}^{\infty} |f(y)|\,|g(x-y)|^2\,dy\right)dx$$

$x-y=t$

$$= \int_{-\infty}^{\infty} \|f\|_1 \left(\int_{-\infty}^{\infty} |f(y)|\,|g(t)|^2\,dt\right)dy = \|f\|_1^2\,\|g\|_2^2$$

$$\Rightarrow \|f*g\|_2 \leq \|f\|_1\|g\|_2 \qquad \blacksquare$$

Ejercicio 2.2.3 $f \in L^\infty(\mathbb{R}) \Leftrightarrow \|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)| < \infty$

Prueba que si $f, g \in L^2(\mathbb{R})$, entonces $f*g \in L^\infty(\mathbb{R})$ y

$\|f*g\|_\infty \leq \|f\|_2\|g\|_2$. (Cauchy-Schwarz)

Para entregar el 12/07 (Segunda feira)