2. Muestreo de señales e imágenes

2.1. TRANSFORMADA DE FOURIER EN $L^1(\mathbb{R})$

- $L^1(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} : f \text{ medibles}, \|f\|_1 = \int_{-\infty}^{\infty} |f(x)|dx < \infty \}$

es un espacio normado y completo. Las funciones de $L^1(\mathbb{R})$ están definidas en casi todo punto

- $L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} : f \text{ medibles}, \|f\|_2 = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right)^{1/2} < \infty \}$

es un espacio normado y completo, con producto escalar

$$\langle f, g \rangle_2 = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx$$

Ejercicio 2.1.1 (a) Sea $f(x) = \begin{cases} 1/\sqrt{x} & \text{si } 0 < x \leq 1 \\ 0 & \text{resto} \end{cases}$. Probar que $f \in L^1(\mathbb{R})$, pero $f \notin L^2(\mathbb{R})$

(b) Sea $g(x) = \begin{cases} 1/x & \text{si } 1 \leq x < \infty \\ 0 & \text{resto} \end{cases}$. Probar que $g \in L^2(\mathbb{R})$, pero $g \notin L^1(\mathbb{R})$

S/ (a) $\int_{-\infty}^{\infty} |f(x)|dx = \int_0^1 x^{-1/2} dx = \left[\frac{x^{1/2}}{1/2}\right]_0^1 = 2 < \infty$

$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_0^1 \frac{1}{x} dx = [\ln x]_0^1 = \infty$

(b) $\int_{-\infty}^{\infty} |f(x)|dx = \int_1^{\infty} \frac{1}{x} dx = [\ln x]_1^{\infty} = \infty$

$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1 < \infty$

Nota: Si $A \subset \mathbb{R}$ con medida finita, $\chi_A(x) = \begin{cases} 1 & \text{si } x \in A \\ 0 & \text{si } x \notin A \end{cases}$ $\in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ porque $\int_{\mathbb{R}} |\chi_A(x)| dx = \int_A dx = |A| < \infty$

y $\int_{\mathbb{R}} |\chi_A(x)|^2 dx = \int_A 1^2 dx = |A| < \infty$

Def 2.1.1. Sea $f \in L^1(\mathbb{R})$; su transformada de Fourier es

$$\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \omega}\, dx.$$

$\mathcal{F}f$ está bien definida: $|\mathcal{F}f(\omega)| = \left|\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \omega}\, dx\right|$ $\leq \int_{-\infty}^{\infty} |f(x)|\, dx = \|f\|_1 < \infty$. Esto es

$$\|\mathcal{F}f\|_\infty := \sup_{\omega \in \mathbb{R}} |\mathcal{F}f(\omega)| \leq \|f\|_1$$

Además, $\mathcal{F}f$ es continua porque

$$\lim_{\omega \to \omega_0} \mathcal{F}f(\omega) = \lim_{\omega \to \omega_0} \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \omega}\, dx \overset{\text{TCDL}}{=}$$

$x \to f(x)\, e^{-2\pi i x \omega} \in L^1(\mathbb{R})$

$$= \int_{-\infty}^{\infty} f(x) \left(\lim_{\omega \to \omega_0} e^{-2\pi i x \omega}\right) dx = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \omega_0}\, dx = \mathcal{F}f(\omega_0)$$

Algunos libros ponen $\hat{f}(\omega) = \mathcal{F}f(\omega)$

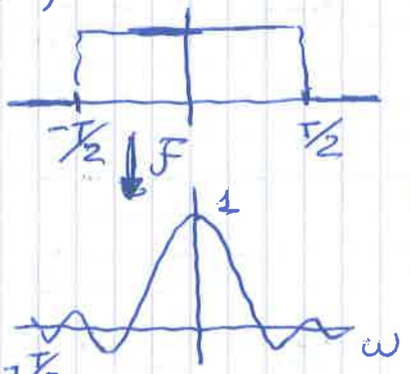
TCDL: Si $\{f_n\}_{n=1}^{\infty}$ y $|f_n(x)| \leq g(x)$ con $g \in L^1(\mathbb{R})$,

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)\, dx = \int_{\mathbb{R}} \left(\lim_{n \to \infty} f_n(x)\right) dx$$

Ejercicio 2.1.2. Sea $f(x) = \frac{1}{T} \chi_{[-\frac{T}{2}, \frac{T}{2}]}(x)$, $T > 0$

Pruebe que

$$\mathcal{F}f(\omega) = \begin{cases} \dfrac{\operatorname{sen} \pi \omega T}{\pi \omega T} & \text{si } \omega \neq 0 \\ 1 & \text{si } \omega = 0 \end{cases}$$



$(\omega \neq 0)$

S/

$$\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \omega}\, dx =$$

$$= \int_{-T/2}^{T/2} \frac{1}{T}\, e^{-2\pi i x \omega}\, dx = \frac{-1}{T} \left[\frac{1}{2\pi i \omega}\, e^{-2\pi i x \omega}\right]_{-T/2}^{T/2}$$

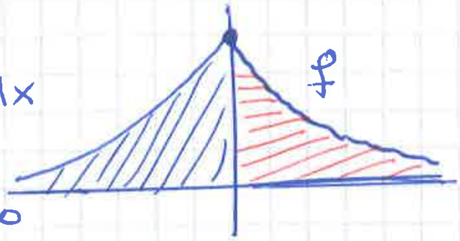
$$= +\frac{1}{T\, 2\pi i \omega} \left[e^{+\pi i \omega T} - e^{-\pi i \omega T}\right] = \frac{1}{\cancel{\pi} \cancel{T} \omega T} \operatorname{sen}(\pi \omega T) = \frac{\operatorname{sen}(\pi \omega T)}{\pi \omega T}$$

Ejercicio 2.1.3. (a) Si $f = \chi_{[a,b]}$, calcular $\mathcal{F}f(\omega)$ y probar que $\lim_{|\omega|\to\infty} \mathcal{F}f(\omega) = 0$

(b) Si $f = \sum_{i=1}^{n} \alpha_i \chi_{[a_i, b_i]}$ (función simple), probar que $\lim_{|\omega|\to\infty} \mathcal{F}f(\omega) = 0$

(Para entregar el 12/07 - Segunda fecha)

Ejercicio 2.1.4. Sea $a>0$. Probar que $f(x) = e^{-a|x|} \in L^1(\mathbb{R})$ y calcular $\mathcal{F}f(\omega)$



S/ $\int_{-\infty}^{\infty} |f(x)|dx = \int_{-\infty}^{0} e^{+ax}dx + \int_{0}^{\infty} e^{-ax}dx$

$= 2\int_0^\infty e^{-ax}dx = -\frac{2}{a}\left[e^{-ax}\right]_0^\infty = \frac{2}{a} < \infty$

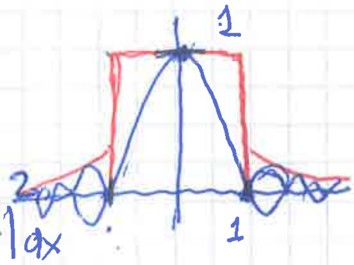
$\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} e^{-a|x|} e^{-2\pi i x\omega} dx = \int_{-\infty}^{0} e^{ax} e^{-2\pi i x \omega} dx$

$+ \int_0^\infty e^{-ax} e^{-2\pi i x\omega} dx = \int_{-\infty}^{0} e^{(a-2\pi i\omega)x} dx + \int_0^\infty e^{(-a-2\pi i\omega)x} dx$

$= \frac{1}{a-2\pi i\omega}\left[e^{(a-2\pi i\omega)x}\right]_{-\infty}^{0} + \frac{1}{-a-2\pi i\omega}\left[e^{(-a-2\pi i\omega)x}\right]_0^\infty$

$= \frac{1}{a-2\pi i\omega} + \frac{1}{a+2\pi i\omega} = \frac{2a}{a^2-(2\pi i\omega)^2} = \frac{2a}{a^2+4\pi^2\omega^2}$

Ejercicio 2.1.5. Probar que la función $\varphi(x) = \begin{cases} \frac{\operatorname{sen}\pi x}{\pi x}, & x\neq 0 \\ 1 & \text{si } x=0 \end{cases}$ es de $L^2(\mathbb{R})$.



$\lim_{x\to 0} \frac{\operatorname{sen}\pi x}{\pi x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{\cancel{\pi}\cos\pi x}{\cancel{\pi}} = 1$

$\int_{-\infty}^{\infty} |\varphi(x)|^2 dx = 2\int_0^\infty |\varphi(x)|^2 dx \le 2\int_0^\infty \left|\frac{\operatorname{sen}\pi x}{\pi x}\right|dx$

$\le 2\int_0^1 1\,dx + 2\int_1^\infty \frac{1}{\pi^2 x^2}dx = 2 + \left[-\frac{2}{\pi^2}\frac{1}{x}\right]_1^\infty = 2 + \frac{2}{\pi^2} < \infty.$

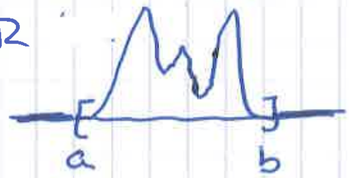
Prop. 2.1.2. (Derivadas y transformada de Fourier)

(a) Si $x^k f(x) \in L^1(\mathbb{R})$ para $k=0,1,2,\dots,n$ entonces $\mathcal{F}f \in C^n(\mathbb{R})$ y $\dfrac{d^k \mathcal{F}f}{d\omega^k}(\omega) = \mathcal{F}\left((-2\pi i x)^k f\right)(\omega)$ $k=1,2,\dots n$

(b) Si $f \in C^n(\mathbb{R}) \cap L^1(\mathbb{R})$ y $f^{(k)} \in L^1(\mathbb{R})$, $k=1,2,\dots,n$, entonces $\mathcal{F}\left(f^{(k)}\right)(\omega) = (2\pi i \omega)^k \mathcal{F}f(\omega)$

(c) Si $f \in L^1(\mathbb{R})$ y tiene soporte compacto, $\mathcal{F}f \in C^\infty(\mathbb{R})$

- f tiene soporte compacto si $\exists\, [a,b] \subset \mathbb{R}$ tal que $f(x)=0$ si $x \notin [a,b]$



D/ (a)

$$\frac{d^k \mathcal{F}f}{d\omega^k}(\omega) = \frac{d^k}{d\omega^k}\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\omega}\,dx = \int_{-\infty}^{\infty} f(x)\,\frac{d^k}{d\omega^k}\left(e^{-2\pi i x\omega}\right)dx$$

$$= \int_{-\infty}^{\infty} \underbrace{f(x)\,(-2\pi i x)^k}\, e^{-2\pi i x\omega}\,dx = \mathcal{F}\left((-2\pi i x)^k f\right)(\omega)$$