09/07/2021

Procesamiento de señales y ondículas

1.6. SERIES DE FOURIER (Continuación)

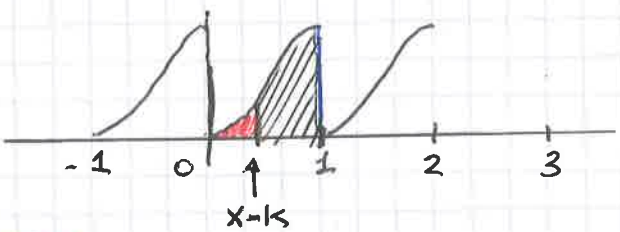
Ejercicio 1.6.3: $S_N f(x) = \int_0^1 f(t) D_N(x-t)\,dt$ donde

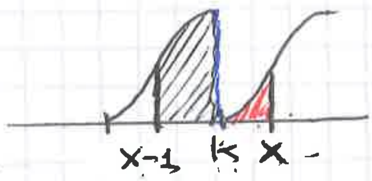
$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi i k t} = \begin{cases} 2N+1 & \text{si } t=0 \\ \frac{\operatorname{sen}(2N+1)\pi t}{\operatorname{sen}\pi t} & \text{si } t\neq 0 \end{cases}$$

es el núcleo de Dirichlet.

Hemos visto que $\int_0^1 D_N(t)\,dt = \sum_{k=-N}^{N} \int_0^1 e^{2\pi i k t}\,dt = 1.$

Recuerda que $f:\mathbb{R}\to\mathbb{C}$ es 1-periódica si $f(x+1)=f(x)$ $\forall x\in\mathbb{R}$. Por inducción $f(x+k)=f(x)$ $\forall k\in\mathbb{Z}$





Ejercicio 1.6.4: Probar que para todo $x\in\mathbb{R}$, si f es 1-periódica,

$$\int_{x-1}^{x} f(t)\,dt = \int_0^1 f(t)\,dt$$

S/ $\exists\, k\in\mathbb{Z}$ t.q. $x-1\leq k< x$ (e.d. $k=[x]$ = parte entera)

$$\int_{x-1}^{x} f(t)\,dt = \int_{x-1}^{k} f(t)\,dt + \int_{k}^{x} f(t)\,dt$$

$t=s+k-1$ $\qquad$ $t=s+k$

$$= \int_{x-k}^{1} f(s+k-1)\,ds + \int_0^{x-k} f(s+k)\,ds$$

f es 1-per

$$= \int_{x-k}^{1} f(s)\,ds + \int_0^{x-k} f(s)\,ds = \int_0^1 f(s)\,ds$$

"Si una función es 1-periódica, $\int_a^b f(s)\,ds = \int_0^1 f(s)\,ds$ si $[a,b]$ tiene longitud 1"

②

$$S_N f(x) = \int_0^1 f(t) D_N(x-t)\,dt \overset{x-t=s}{=\!=} -\int_x^{x-1} f(x-s) D_N(s)\,ds$$

$$= \int_{x-1}^{x} \underbrace{f(x-s)}_{1\text{-per}} \underbrace{D_N(s)}_{1\text{-per}}\,ds \overset{\text{Ej } 1.6.3}{=} \int_0^1 f(x-s) D_N(s)\,ds$$

Lema 1.6.2. (Riemann-Lebesgue). Si $f \in L^1_p([0,1])$, se tiene $\lim_{|k|\to\infty} \hat{f}(k) = 0$

P/ $e^{-\pi i} = \cos\pi - i \operatorname{sen}\pi = \cos\pi = -1$

$$\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x}\,dx = -\int_0^1 f(x) e^{-2\pi i k x} e^{-\pi i}\,dx$$

$$= -\int_0^1 f(x) e^{-2\pi i k (x + \frac{1}{2k})}\,dx \overset{x+\frac{1}{2k}=y}{=\!=} -\int_{1/2k}^{1+\frac{1}{2k}} f(y - \tfrac{1}{2k}) e^{-2\pi i k y}\,dy$$

$$\left\{\begin{matrix} f \text{ es } 1\text{-per} \\ e^{-2\pi i k y} \text{ es } 1\text{-per} \end{matrix}\right\} = -\int_0^1 f(y - \tfrac{1}{2k}) e^{-2\pi i k y}\,dy.$$

$$2\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x}\,dx - \int_0^1 f(x - \tfrac{1}{2k}) e^{-2\pi i k x}\,dx$$

$$= \int_0^1 [f(x) - f(x - \tfrac{1}{2k})] e^{-2\pi i k x}\,dx$$

Si f es continua

$$0 \le \lim_{|k|\to\infty} |\hat{f}(k)| \le \frac{1}{2} \int_0^1 \lim_{|k|\to\infty} |f(x) - f(x - \tfrac{1}{2k})|\,dx = 0.$$

Como $C([0,1])$ son densos en $L^1_p([0,1])$, dado $\varepsilon > 0$, $\exists$ $g \in C([0,1])$ t.q. $\|f - g\|_1 < \varepsilon/2$. Como g es continua, $\exists |k|$ grande t.q. $|\hat{g}(k)| < \varepsilon/2$. Entonces,

$$|\hat{f}(k)| = |\hat{f}(k) - \hat{g}(k) + \hat{g}(k)| \overset{DT}{\le} |(f-g)^\wedge(k)| + |\hat{g}(k)|$$

$$= \left| \int_0^1 (f(x) - g(x)) e^{-2\pi i x k}\,dx \right| + |\hat{g}(k)| \le \int_0^1 |f(x) - g(x)|\,dx + |\hat{g}(k)|$$

$$= \|f - g\|_1 + |\hat{g}(k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{si } |k| \text{ grande.}$$

③

Si $f \in L^1_p([0,1])$ definir

$$\sigma_n f(x) = \frac{1}{N} \sum_{k=0}^{N-1} S_k f(x)$$

Tenemos

$$\sigma_n f(x) = \frac{1}{N} \sum_{k=0}^{N-1} \int_0^1 f(x-t) D_k(t) dt = \int_0^1 f(x-t) \underbrace{\left[\frac{1}{N} \sum_{k=0}^{N-1} D_k(t)\right]}_{F_N(t)} dt$$

$$:= \int_0^1 f(x-t) F_N(t) dt$$

donde $F_N(t) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(t)$ es el núcleo de Fejér

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Ejercicio 1.6.5. Probar que el núcleo de Fejér es 1-periódica, par y

$$F_N(t) = \begin{cases} \frac{1}{N} \left(\frac{\operatorname{sen} \pi N t}{\operatorname{sen} \pi t} \right)^2 & \text{si } t \in \mathbb{R} \setminus \mathbb{Z} \\ N & \text{si } t \in \mathbb{Z} \end{cases}$$

S/ Basta probarlo en $t \in [-\frac{1}{2}, \frac{1}{2}]$. Si $t=0$

$$F_N(0) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(0) = \frac{1}{N} [1+3+5+\dots+(2N-1)] = \frac{1}{\cancel{N}} \frac{\cancel{2} N \cdot N}{\cancel{2}} = N$$

Si $t \neq 0$

$$F_N(t) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{\operatorname{sen} \pi(2k+1)t}{\operatorname{sen}(\pi t)} = \frac{1}{N} \frac{1}{\operatorname{sen}(\pi t)} \underbrace{\sum_{k=0}^{N-1} \operatorname{sen}(2k+1)\pi t}_{(*)}$$

- $\cos(A-B) - \cos(A+B) = 2 \operatorname{sen} A \operatorname{sen} B$

$A = (2k+1)\pi t$

$B = \pi t$

$$(*) = \sum_{k=0}^{N-1} \operatorname{sen}(2k+1)\pi t =$$

$$= \frac{1}{2} \sum_{k=0}^{N-1} \frac{1}{\operatorname{sen} \pi t} [\cos(2k\pi t) - \cos 2(k+1)\pi t]$$

$$= \frac{1}{2 \operatorname{sen} \pi t} [1 - \cancel{\cos 2\pi t} + \cancel{\cos 2\pi t} - \cancel{\cos 4\pi t} - \dots - \cos 2N\pi t]$$

$$= \frac{1}{2 \operatorname{sen} \pi t} [1 - \cos(2N\pi t)] = \frac{\operatorname{sen}^2 \pi N t}{\operatorname{sen} \pi t}$$

$$\left(\frac{1 - \cos 2\alpha}{2} = \operatorname{sen}^2 \alpha \right)$$

Clase mañana: 8:30 am - 11:30 am con descanso

Test 1: 14/07 (Cuarta feira) 8:00 pm - 9:30 pm

Cap 1

Test 2: 21/07 (Cuarta feira) 8:00 pm - 9:30 pm

$$\int_{-1/2}^{1/2} |F_N(t)|\,dt = \int_{-1/2}^{1/2} F_n(t)\,dt = \int_{-1/2}^{1/2} \left(\frac{1}{N}\sum_{k=0}^{N-1} D_k(t)\right)dt$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \underbrace{\int_{-1/2}^{1/2} D_k(t)\,dt} = \frac{1}{N}\sum_{k=0}^{N-1} 1 = 1$$

Lema 1.6.3 Para todo $\delta > 0$, $\lim_{N\to\infty} \int_{\delta<|t|<\frac{1}{2}} F_N(t)\,dt = 0$

Teorema 1.6.4. Si $f \in L_p^2([-\frac{1}{2}, \frac{1}{2}])$, entonces $\lim_{N\to\infty} \|\sigma_N f - f\|_2 = 0$

P/ Como $\int_{-1/2}^{1/2} F_N(t)\,dt = 1$, se tiene

$$\sigma_N f(x) - f(x) = \int_{-1/2}^{1/2} f(x-t)\,F_N(t)\,dt - \int_{-1/2}^{1/2} f(x)\,F_N(t)\,dt$$

$$= \int_{-1/2}^{1/2} [f(x-t) - f(x)]\,F_N(t)\,dt$$

Entonces,

$$\|\sigma_N f - f\|_{L^2([-\frac{1}{2},\frac{1}{2}])} = \left\| \int_{-1/2}^{1/2} [f(x-t) - f(x)]\,F_N(t)\,dt \right\|_{L^2([-\frac{1}{2},\frac{1}{2}])} \overset{\text{Minkowski}}{=}$$

$$\leq \int_{-1/2}^{1/2} \|f(\cdot - t) - f(\cdot)\|_{L^2([-\frac{1}{2},\frac{1}{2}])}\,F_N(t)\,dt =$$

$$= \int_{|t|<\delta} \|f(\cdot - t) - f(\cdot)\|_2\,F_N(t)\,dt + \int_{\delta<|t|<1/2} \|f(\cdot - t) - f(\cdot)\|_2\,F_N(t)\,dt$$

$$= I_{\delta,N} + II_{\delta,N}$$

Como $\lim_{t\to 0} \|f(\cdot - t) - f(\cdot)\|_2 = 0$, dado $\varepsilon > 0$, existe $\delta_0 < \frac{1}{2}$ tal que $\|f(\cdot - t) - f(\cdot)\|_2 < \varepsilon/2$ si $|t| < \delta_0$. Entonces,

$$I_{\delta_0, N} \leq \frac{\varepsilon}{2} \int_{|t| \leq \delta_0} F_N(t)\,dt \leq \frac{\varepsilon}{2} \int_{-1/2}^{1/2} F_N(t)\,dt = \varepsilon/2.$$

Para δ_0, por el lema 1.6.3, $\exists\, N_0 \in \mathbb{N}$ tal que

$$\int_{\delta_0 < |t| < 1/2} F_N(t)\,dt \leq \frac{\varepsilon}{4\|f\|_2}, \quad \forall N \geq N_0$$

Además,

$$\|f(\cdot - t) - f(\cdot)\|_2 \overset{DT}{\leq} \|f(\cdot - t)\|_2 + \|f\|_2 = \left(\int_{-1/2}^{1/2} |f(x-t)|^2 dx \right)^{1/2} + \|f\|_2 \overset{x-t=y}{=} \left(\int_{-1/2 - t}^{1/2 - t} |f(y)|^2 dy \right)^{1/2} + \|f\|_2 = \left(\int_{-1/2}^{1/2} |f(y)|^2 dy \right)^{1/2} + \|f\|_2$$

$$= 2\|f\|_2$$

Ahora

$$II_{\delta_0, N} \leq 2\|f\|_2 \int_{\delta_0 < |t| < 1/2} F_N(t)\,dt \leq 2\|f\|_2 \cdot \frac{\varepsilon}{4\|f\|_2} = \frac{\varepsilon}{2}, \quad N \geq N_0.$$

Entonces

$$\|\sigma_N f - f\|_2 \leq I_{\delta_0, N} + II_{\delta_0, N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{si } N \geq N_0$$

Corolario 1.6.5. El conjunto $\{e_n(x) = e^{2\pi i n x} : n \in \mathbb{Z}\}$ es una <u>base</u> o.n. de $L^2([-1/2, 1/2])$ ó $L^2([0,1])$

D/ Como $\{e_n(x) = e^{2\pi i n x} : n \in \mathbb{Z}\}$ es s.o.n, falta probar que es completo, e.d. $\text{Span}\{e_n(x)\}_{n=1}^{\infty}$ es denso en $L^2([-1/2, 1/2])$. Si $f \in L^2([-1/2, 1/2])$, por el Teorema 1.6.4, dado $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ tal que $\|\sigma_N f - f\|_2 < \varepsilon$.

Pero $\sigma_N f(x) = \frac{1}{N} \sum_{k=0}^{N-1} S_k f(x) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=-k}^{k} \hat{f}(j)\, e^{2\pi i j x}$

$= \sum_{k=0}^{N-1} \sum_{j=-k}^{k} \frac{1}{N} \hat{f}(j)\, e^{2\pi i j x}$ es un elemento de span$\{e_n\}_{n\in\mathbb{Z}}$.

Por tanto, span$\{e_n\}_{n\in\mathbb{Z}}$ es denso en $L^2([-\frac{1}{2},\frac{1}{2}])$.

Ejercicio 1.6.6 (Para entregar el 12/07 - Segunda feira)

Probar que $\{e_n(x) = \frac{1}{\sqrt{T}} e^{2\pi i \frac{n}{T} x} : n \in \mathbb{Z}\}$ es completo en $L^2([a,b])$ con $T = b-a$.

Sugerencia: Probar que se cumple la identidad de Plancherel para $f \in L^2([a,b])$ usando que se cumple para $L^2([0,1])$.
