Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2020-21 Homework 4
Due: Wednesday, May 12, 2021

Explicit constructions of wavelets.

1. A scaling function of the Haar MRA is $\varphi = \chi_{[0,1]}$ and $\mathcal{F}\varphi(\xi) = e^{-\pi i \xi}\dfrac{\sin(\pi\xi)}{\pi\xi}$. The low pass filter of the Haar MRA is $h(\xi) = e^{-\pi i\xi}\cos(\pi\xi)$. Therefore, the following formula must hold:

$$\prod_{j=1}^{\infty} e^{-\pi i\xi/2^j}\cos(\frac{\pi\xi}{2^j}) = e^{-\pi i\xi}\frac{\sin(\pi\xi)}{\pi\xi}.$$

Show this formula directly using the trigonometric relation $\sin(2\alpha) = 2(\sin\alpha)(\cos\alpha)$.

2. Let $h(\xi) = e^{-3\pi i\xi}\cos(3\pi\xi)$. Clearly, h is a continuous function on $\mathbb{R}$, it is one periodic, and $h(0) = 1$.

a) Show that

$$|h(\xi)|^2 + h(\xi + 1/2)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}.$$

b) Define φ by $\mathcal{F}\varphi(\xi) = \prod_{j=1}^{\infty} h(\xi/2^j)$. Show that $\varphi = \dfrac{1}{3}\chi_{[0,3]}$. (Observe that the integer translates of φ do not form an orthogonal system in $L^2(\mathbb{R})$.)

3. Let c_k be given by

$$\frac{1}{c_k} = \int_0^{1/2} (\sin 2\pi x)^{2k+1} dx > 0,$$

and

$$g_k(\xi) = 1 - c_k \int_0^{\xi} (\sin 2\pi x)^{2k+1} dx\,.$$

a) Show that

$$g_0(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^2.$$

b) Show that

$$g_1(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^4 (2 - \cos 2\pi\xi)\,.$$

(Hint: Write $\sin^2 x = (1 + \cos x)(1 - \cos x)$ and integrate by parts.)

4. (2 puntos) Find the coefficients of a low pass filter of Daubechies ${}_2\psi$ orthonormal wavelet using the polynomial

$$g_1(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^4 (2 - \cos 2\pi\xi)\,,$$

found in the previous exercise.

5. (2 puntos) Let c_k be given by

$$\frac{1}{c_k} = \int_0^{1/2} (\sin 2\pi x)^{2k+1} dx > 0,$$

and

$$g_k(\xi) = 1 - c_k \int_0^{\xi} (\sin 2\pi x)^{2k+1} dx \,.$$

Show, integrating by parts, that

$$g_k(\xi) = \left|\frac{1 + e^{-2\pi i \xi}}{2}\right|^{2k+2} P_k(\xi) \,,$$

where

$$P_k(\xi) = \frac{1}{2^k} \sum_{\ell=0}^{k} \binom{2k+1}{k-\ell} (1 + \cos 2\pi\xi)^{\ell} (1 - \cos 2\pi\xi)^{k-\ell}$$

is an even trigonometric polynomial with real coefficients.