Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2020-21 Homework 3
Due: Monday, April 26, 2021

Wavelets and Multiresolution Analysis.

1. Let $\psi \in Ł^2(\mathbb{R}) \setminus 0$. For $f \in L^2(\mathbb{R})$ the wavelet transform of f is defined by

$$W_\psi(f)(y,s) = \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{s}} \overline{\psi(\frac{x-y}{s})}\, dx\,, \quad y \in \mathbb{R}\,, s > 0\,.$$

Suppose that

$$\int_0^\infty \frac{|\mathcal{F}(\psi)(\xi)|^2}{\xi}\, d\xi = C_\psi = \int_0^\infty \frac{|\mathcal{F}(\psi)(-\xi)|^2}{\xi}\, d\xi < \infty$$

Prove the following reconstruction formula for $f \in L^2(\mathbb{R})$:

$$f(x) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{\infty} W_\psi(f)(y,s) \frac{1}{\sqrt{s}} \psi(\frac{x-y}{s})\, dy\, \frac{ds}{s^2}\,. \tag{1}$$

(Hint: Prove that the right hand side of (1) conicides with

$$b(x) = \frac{1}{C_\psi} \int_0^\infty f * D_{1/s}\widetilde{\psi} * D_{1/s}\psi(x) \frac{ds}{s^2}$$

where $\widetilde{\psi} = \overline{\psi(-x)}$. Then prove that b and f have the same Fourier transform.)

2. Find the Haar coefficients, that is, $< f, \psi_{j,k} >$ for all $j,k \in \mathbb{Z}$, for the function $f = \chi_{[0,1)}$, where ψ is the Haar wavelet.

3. Show that for the Haar-MRA with scaling function $\varphi = \chi_{[0,1)}$, its low pass filter coefficients are given by $h[0] = h[1] = 1/2$ and $h[k] = 0$ if $k \neq 0,1$, and the low pas filter is $h(\xi) = e^{-\pi i \xi}\cos(\pi\xi)$.

4. Show that for the Shannon-MRA with scaling function φ given by $\varphi(x) = \dfrac{\sin(\pi x)}{\pi x}\,, x \neq 0$ and $\varphi(0) = 1$ (recall that $\mathcal{F}\varphi = \chi_{[-1/2,1/2)}$) the associate discrete filter is

$$h[k] = \begin{cases} 0 & \text{if } k = 2\ell, \ell \neq 0 \\ 1/2 & \text{if } k = 0 \\ \frac{(-1)^\ell}{\pi(2\ell+1)} & \text{if } k = 2\ell+1 \end{cases}$$

Show that the low pass filter of this MRA is

$$h(\xi) = \sum_{k=-\infty}^{\infty} \chi_{[-\frac{1}{4},\frac{1}{4})}(\xi + k)\,.$$

5. Let $h(\xi)$ and $g(\xi)$ be two 1-periodic functions in $L^2([0,1))$ satisfying $g(\xi) = e^{-2\pi i\xi}\overline{h(\xi + \frac{1}{2})}$. If $h(\xi) = \sum_{k=-\infty}^{\infty} h[k]e^{-2\pi ik\xi}$ and $g(\xi) = \sum_{k=-\infty}^{\infty} g[k]e^{-2\pi ik\xi}$, show that

$$g[k] = \overline{h[1-k]}(-1)^{1-k}\,, \quad k \in \mathbb{Z}\,.$$

6. Let $(\{V_j\}_{j\in\mathbb{Z}}), \varphi)$ be an MRA of $L^2(\mathbb{R})$. Consider the wavelet ψ given by Mallat's recipe wiht $\nu(\xi) = 1, \xi \in \mathbb{R}$, that is,

$$\mathcal{F}\psi(\xi) = g(\frac{\xi}{2})\mathcal{F}\varphi(\frac{\xi}{2}) \qquad \text{and} \qquad g(\xi) = e^{-2\pi i\xi}\overline{h(\xi + \frac{1}{2})}\,,$$

where $h(\xi)$ is the low-pass filter associated to the MRA.

Prove that

$$\psi(x) = 2\sum_{k=-\infty}^{\infty}(-1)^{1-k}\overline{h[1-k]}\varphi(2x-k)\,.$$

7. Consider the Haar-MRA with scaling function $\varphi = \chi_{[0,1)}$. Show that an orthonormal wavelet associated to this MRA is given by $\psi(x) = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}$.

8. For the Shannon MRA with scaling function $\mathcal{F}\varphi = \chi_{[-\frac{1}{2},\frac{1}{2})}$, show that one of this associated orthonormal wavelets is given by

$$\mathcal{F}\psi(\xi) = e^{-\pi i\xi}\chi_{[-1,-\frac{1}{2})\cup[\frac{1}{2},1)}\,.$$

9. Show that the detail coefficients $d_{j-1}(p)$ are computed with the formula

$$d_{j-1}(p) = \sqrt{2}\sum_{k\in\mathbb{Z}}\overline{g[k-2p]}c_j(k)\,,$$

where $\{c_j(k) : k \in \mathbb{Z}\}$ are the coefficients of f at level j and $\{g(k) : k \in \mathbb{Z}\}$ are the high pass filter coefficients.

10. For the Haar wavelet, show that the decomposition algorithm is

$$c_{j-1}(p) = \sqrt{2}\,\frac{c_j(2p) + c_j(2p+1)}{2}\,, \quad d_{j-1}(p) = \sqrt{2}\,\frac{-c_j(2p) + c_j(2p+1)}{2}\,,$$

and the reconstruction algorithm is

$$c_j(2p) = \sqrt{2}\,\frac{c_{j-1}(p) - d_{j-1}(p)}{2}\,, \quad c_j(2p+1) = \sqrt{2}\,\frac{c_{j-1}(p) + d_{j-1}(p)}{2}\,.$$