Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2020-21 Homework 2-2
Due: Thursday, March 18, 2021

Orthonormal bases for signal and image processing.Part II

1. Show that the N vectors given by

$$\mu_k\sqrt{\frac{2}{N-1}}\Big(\lambda_n \cos[\frac{\pi}{N-1}\,k\,n]\Big)_{n=0}^{N-1}, \qquad k=0,1,2,\dots,N-1\,,$$

each one of size N, where $\lambda_0 = 1/\sqrt{2}$, $\lambda_{N-1} = 1/\sqrt{2}$ and $\lambda_n = 1$ if $n = 1,2,\dots,N-2$, and $\mu_0 = \mu_{N-1} = 1/\sqrt{2}$, and $\mu_k = 1$ if $k = 1,2,\dots,N-2$, is an orthonormal basis of the space of signals of size N. This basis corresponds to the one obtained by extending $f = (f(n))_{n=0}^{N-1}$ evenly with respect to $n = 0$.

2. (20 points) This exercise shows how to calculate DCT-I with an induction relation that involves DCT-IV.

a) Regroup the terms $f(n)$ and $f(N-1-n)$, $0 \le n \le \dfrac{N}{2}-1, N = 2^q$, in the DCT-I, to write $\widehat{f}_I(2k)$ as the DCT-I of the signal

$$s(n) = \frac{1}{\sqrt{2}}[f(n) + f(N-1-n)], \qquad 0 \le n \le \frac{N}{2} - 1.$$

b) With the same technique as in part a), write $\widehat{f}_I(2k+1)$ as the DCT-IV of the signal

$$r(n) = \frac{1}{\sqrt{2}}[f(n) - f(N-1-n)], \qquad 0 \le n \le \frac{N}{2} - 1.$$

c) Using that, with a fast algorithm, the number of operations to calculate DCT-IV of size N is $O(N\log_2 N)$ and parts a) and b), show that with the above algorithm, the number of operations needed to calculate DCT-I of size N is also $O(N\log_2 N)$.

3. Show that the number $B_j^{(2)}$ of orthogonal bases of the space of discrete images of size $N^2 (N = 2^L)$ in a bi-dyadic tree of depth $j, 1 \le j \le L$, satisfies

$$2^{4^{j-1}} \le B_j^{(2)} \le 2^{\frac{4}{3}4^{j-1}}.$$

4. Consider the signal f of size $N = 8$ given by

$$f = (8, 16, 24, 32, 40, 48, 56, 64)\,.$$

a) Compute the DCT-I of f, rounding the result to the nearest integer. Compress the signal 50% by setting to zero the DCT-I coefficients in positions 4, 5, 6, and 7. Find now the inverse DCT-I of this compressed signal, and, after rounding, observe that is similar to the original one.

b) Take now the orthonormal basis of $\mathbb{C}^8$ given by

$$\left\{\lambda_k \frac{1}{2}\left(\cos\frac{\pi k n}{4}\right)_{n=0}^{7}\right\}_{k=0}^{4} \bigcup \left\{\frac{1}{2}\left(\sin\frac{\pi k n}{4}\right)_{n=0}^{7}\right\}_{k=1}^{3}.$$

where $\lambda_0 = \lambda_4 = \dfrac{1}{\sqrt{2}}$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Repeat the process in a), setting now to zero the frequencies $k = 3$ and $k = 4$ of cosines, and the frequencies $k = 2$ and $k = 3$ of sines. Observe that the final result is somehow different than the original signal.