Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2020-21 Homework 2-1
Due: Wednesday, March 10, 2021

Orthonormal bases for signal and image processing.Part I

1. Given $f : [0,1] \longrightarrow \mathbb{R}$ with $f \in L^2([0,1])$, extend f to $\mathbb{R}$ to obtain a function $\widetilde{f}$ odd with respect to the origin, even with respect to 1 and -1 and 4-periodic. By writing the Fourier series of $\widetilde{f}$ in $[-2,2]$ in terms of sines and cosines show that the cosine coefficients are zero as well as the even sine coefficients. Prove that

$$\{\sqrt{2}\sin(\frac{2k+1}{2}\pi x) : k = 0,1,2,\dots\}$$

is and orthornormal basis of $L^2([0,1])$. This is called the **sine-IV** basis for $L^2([0,1])$.

2. Show that for a function $f \in L^2(\mathbb{R}) \cap C^2(\mathbb{R})$, the coefficients of f in the block cosine-I basis given by

$$\{\chi_{[n,n+1)}(x) : n \in \mathbb{Z}\} \cup \{\chi_{[n,n+1)}(x)\sqrt{2}\cos \pi k(x-n) : n \in \mathbb{Z}, k = 1,2,\dots\}$$

decay, for n fixed, at a rate proportional at least to $1/k^2$.

3. Given $\varepsilon > 0$, choose ψ an even, C^∞ function defined on $\mathbb{R}$, supported on $[-\varepsilon, \varepsilon]$ such that $\int_\varepsilon^\varepsilon \psi(x) = \pi/2$. Let $\theta(x) = \int_{-\infty}^x \psi(y)dy$. Show that $\theta(x) + \theta(-x) = \pi/2$. Define $s_\varepsilon(x) = \sin\theta(x)$. Show that $[s_\varepsilon(x)]^2 + [s_\varepsilon(-x)]^2 = 1$.

4. With the same notation as in the previous exercise, let $c_\varepsilon(x) = \cos(\theta(x))$. Let $I = [\alpha, \beta] \subset \mathbb{R}, \varepsilon, \varepsilon' > 0$, such that $\alpha + \varepsilon < \beta - \varepsilon'$. The function

$$b_I(x) = s_\varepsilon(x-\alpha)c_{\varepsilon'}(x-\beta)$$

is called a **bell** function associated with the interval $I = [\alpha, \beta]$.

a) Sketch the graph of the bell function b_I.

b) Show that on $[\alpha - \varepsilon, \alpha + \varepsilon]$

- i) $b_I(x) = s_\varepsilon(x-\alpha)$.
- ii) $b_I(2\alpha - x) = s_\varepsilon(\alpha - x) = c_\varepsilon(x-\alpha)$.
- iii) $b_I^2(x) + b_I^2(2\alpha - x) = 1$.

5. Show that the collection of N vectors

$$\mu_k \frac{1}{\sqrt{N}}\Big(\sin\frac{k\pi}{N}(n+\frac{1}{2})\Big)_{n=-N}^{N-1}, \qquad k = 1,2,\dots,N,$$

each one of size $2N$, where $\mu_k = 1$ if $k = 1,2,\dots,N-1$ and $\mu_N = 1/\sqrt{2}$, is an orthonormal system.