

**REPRODUCING SYSTEMS, WAVELETS AND APPLICATIONS  
AFRICAN MATHEMATICAL SCHOOL – CIMPA**

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# REPRODUCING SYSTEMS, WAVELETS AND APPLICATIONS

EMA School - Luanda (Angola)

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22/08/2016

## LECTURE 1

### 1.1. INTRODUCTION

#### 1.1.1. Discovering hydrocarbon fields.

Transparencies of Talk UC3M -2015 - EH

To study a wave  $f(t)$ ,  $0 \leq t \leq 1$ , try to write  $f$  as a superposition of "fundamental waves"

$$f(t) \approx \sum_{k=0}^{\infty} c_k e^{j2\pi k t} + \sum_{k=1}^{\infty} b_k \sin j2\pi k t$$

Since  $e^{jx} = \cos x + j \sin x$  (Euler), one could try

$$f(t) \approx \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t}$$

Since  $\int_0^1 e^{j2\pi k t} e^{-j2\pi l t} dt = \delta_{k,l}$ , it follows that  $c_k = \int_0^1 f(t) e^{-j2\pi k t} dt := \hat{f}(k)$ .

If the wave last for  $T$  seconds, examine in each interval  $[n, n+1]$ ,  $0 \leq n \leq T-1$  the signal  $f \cdot X_{[n, n+1]}$  to write

$$f(t) \approx \sum_{n=0}^{T-1} \sum_{k=-\infty}^{\infty} c_{k,n} e^{j2\pi k t} \cdot X_{[n, n+1]}(t)$$

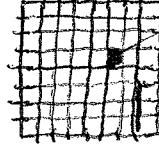
With  $c_{k,n} = \int_n^{n+1} f(t) e^{-j2\pi k t} dt$ . More generally, one can consider systems of the form  $\{e^{j2\pi k t}, g(t-n)\}_{n=0, k \in \mathbb{Z}}^{T-1}$  which

leads to Gabor systems to study singularities of waves.



### 1.1.2. Images: compression and edge detection

Introduction to the article of the Revista de la Unión Matemática Argentina - 2004

 Divide  $[0, 1]^2$  in equal "pixels" of size  $2^{-j}$  to obtain  $I_{k,e}^{(j)} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right] \times \left[\frac{e}{2^j}, \frac{e+1}{2^j}\right]$ ,  $0 \leq k \leq 2^j$ ,  $0 \leq e \leq 2^j$

If the picture is ~~represented~~ given by  $f(x,y)$  on  $[0, 1]^2$  a way to represent  $f$  is to observe the light intensity over each pixel,

$$P_{k,e}^{(j)} = \frac{1}{|I_{k,e}^{(j)}|} \iint_{I_{k,e}^{(j)}} f(x,y) dx dy$$

$$\text{Again, we have } f(x,y) \sim \sum_{k=0}^{2^j} \sum_{e=0}^{2^j} P_{k,e}^{(j)} X_{I_{k,e}^{(j)}}(x,y).$$

One could also consider expansions of the form

$$f(x,y) \sim \sum_{k=0}^{2^j} \sum_{e=0}^{2^j} P_{k,e}^{(j)} \Psi_{k,e}^{(j)}(x,y)$$

with  $\Psi_{k,e}^{(j)}(x) = 2^{\frac{j}{2}} \varphi(2^j(x,y) - (k,e))$  for appropriate  $\varphi$ . This will lead to wavelets

1) Compression: find a good approximation to  $f$  with a small number of coefficients.

2) Edge detection: position of large coefficients will determine the edge of the image

## 1.2 A TOOL: THE FOURIER TRANSFORM IN $\mathbb{R}^n$

Page 4 in [HW] or Section 1.1 of "Notes"

for  $f \in L^1(\mathbb{R}^n)$ ,

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^n$$

$\mathcal{F}: L^1(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)$  with  $\|\mathcal{F}\| \leq 1$  and it is a continuous function by the LDC Theorem.

Ex 1.1. Show that if  $f(x) = \frac{1}{T} \chi_{[-\frac{T}{2}, \frac{T}{2}]}(x)$ ,  $x \in \mathbb{R}$ , then

$$\mathcal{F}f(\xi) = \frac{\text{sinc } \pi T \xi}{\pi T \xi} \quad (\because \text{sinc}(T\xi))$$

Ex 1.2. (\*) Show that if  $f(x) = e^{-4\pi^2 x^2}$ ,  $x \in \mathbb{R}$ , then

$$\mathcal{F}f(\xi) = \frac{1}{2\sqrt{\pi}} e^{-\xi^2/4}, \quad \xi \in \mathbb{R}.$$

"The Fourier transform of two different signals may be similar" (Talk UC3M - 2015)

$\mathcal{F}$  defined on  $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  can be extended to  $L^2(\mathbb{R}^n)$

and  $\mathcal{F}: L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$  with  $\|\mathcal{F}\| = 1$  i.e.,  $\|\mathcal{F}f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$  (Plancherel Thm). The inverse of  $\mathcal{F}$  is

$$\mathcal{F}^{-1} g(x) = \int_{\mathbb{R}^n} g(\xi) e^{2\pi i x \cdot \xi} d\xi$$

Since  $L^2(\mathbb{R}^n)$  is a Hilbert space, polarization gives

$$\langle \mathcal{F}f, \mathcal{F}g \rangle = \langle f, g \rangle \quad \forall f, g \in L^2(\mathbb{R}^n) \quad (\text{Parseval})$$

Poisson Summation Formula (PSF):  $\sum_{k \in \mathbb{Z}^n} f(x+k) = \sum_{k \in \mathbb{Z}^n} Ff(k) e^{2\pi i k \cdot x}$  (PSF-1)

for  $f \in \mathcal{S}(\mathbb{R}^n)$ , and  $\sum_{k \in \mathbb{Z}^n} |f(k)|^2 = \sum_{k \in \mathbb{Z}^n} |Ff(k)|^2$  (PSF-2)

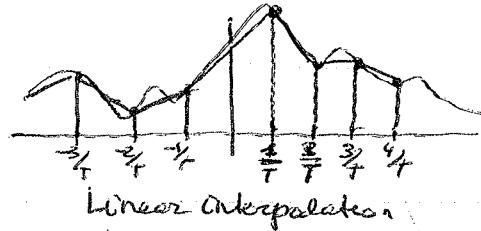
Ex. 1.3

### 1.3. SAMPLING THEOREM

$f(t)$ ,  $t \in \mathbb{R}$ ;  $T > 0$

Sampling rate:  $\frac{1}{T}$

Consider  $\{f(\frac{n}{T})\}_{n \in \mathbb{Z}}$



C. Shannon (1948) - and also Whittaker (1936) and Whittaker-  
proved that if  $\sup Ff \subset [-\frac{T}{2}, \frac{T}{2}]$ , then  $f$  can be recovered  
precisely with samples  $\{f(\frac{n}{T})\}_{n \in \mathbb{Z}}$  by means of combinations  
of superpositions of sinc functions.

Thm 1.1 (Whittaker-Shannon) (Sampling Theorem)

$f \in L^1(\mathbb{R}^n)$  and  $\sup Ff \subset [-\frac{T}{2}, \frac{T}{2}]$ ,  $T > 0$ . Then

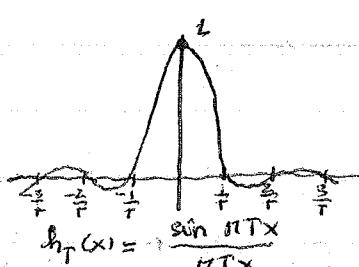
$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) \frac{\sin \pi c(Tx-k)}{\pi c(Tx-k)}, \quad x \in \mathbb{R}. \quad (\text{Convergence in } L^2(\mathbb{R})).$$

P/ By Paley-Wiener,  $f = F h_F$  where  $F$  is entire on  $\mathbb{C}$  of exponential type. It makes sense to consider  $f(\frac{k}{T})$ .

See § 1.2 in my "Notes" or § 6.1. in [HW].

Uses:

- PFS-2;
- Inversion formula;
- Ex. 1.1.



$$\text{Since } \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{2\pi i(x-\frac{k}{T})s} ds = \frac{\sin \pi c(Tx-k)}{\pi c(Tx-k)}$$

$$= h_T(x - \frac{k}{T})$$

(follows from Exercise 1.1, changing  
 $s \leftrightarrow x - \frac{k}{T}$ )

$$\mathcal{F}^{-1}\left(\frac{1}{T} \chi_{[-\frac{T}{2}, \frac{T}{2}]} e^{-2\pi i \frac{k}{T} s}\right) = \frac{\sin \pi c(Tx-k)}{\pi c(Tx-k)} = h_T(x - \frac{k}{T})$$

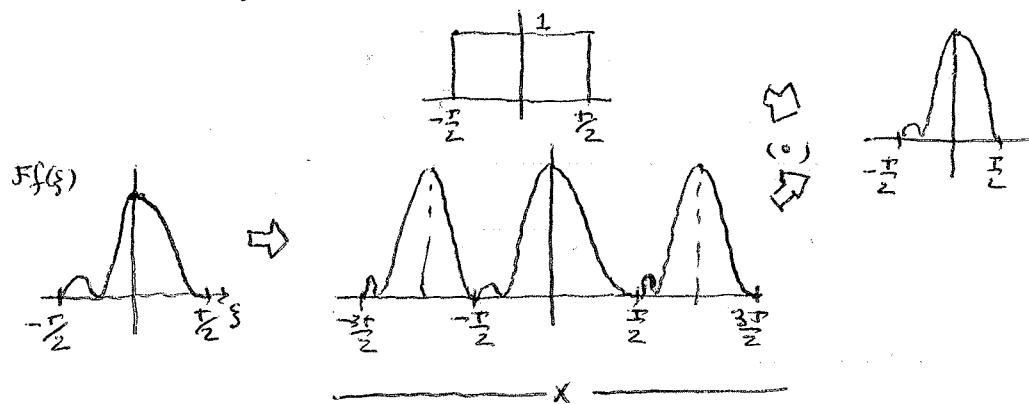
Take Fourier  $\hat{f}$ , in sampling formula:

$$\mathcal{F}f(\xi) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) \mathcal{F}(h_T(\cdot - \frac{k}{T}))(\xi) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) \frac{1}{T} \chi_{[-\frac{T}{2}, \frac{T}{2}]}(\xi) e^{-2\pi i \frac{k}{T} \xi}$$

$$= \frac{1}{T} \left( \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) e^{-2\pi i \frac{k}{T} \xi} \right) \chi_{[-\frac{T}{2}, \frac{T}{2}]}(\xi)$$

(This formula appears in the proof of the sampling theorem)

$$\left( \sum_{k=-\infty}^{\infty} \mathcal{F}f\left(\xi + Tk\right) \right) \chi_{[-\frac{T}{2}, \frac{T}{2}]}(\xi) = \mathcal{F}f(\xi)$$



Explain Aliasing phenomena when sampling rate is bigger than Shannon sampling rate:  $T_0$  s.t.  $\text{supp } f \subset [-\frac{T_0}{2}, \frac{T_0}{2}]$   
(smallest  $T_0$ )

Sampling rate:  $T_s(f) = \text{smallest } T \text{ s.t. } \text{supp } f \subset [-\frac{T}{2}, \frac{T}{2}]$

### Proof of Sampling Theorem

Start

$$\left( \sum_{k=-\infty}^{\infty} Ff(\xi + Tk) \right) \cdot x_{[-\frac{T}{2}, \frac{T}{2}]}(\xi) = Ff(\xi) \quad (1)$$

Consider  $g(x) = \frac{1}{T} f\left(\frac{x}{T}\right)$ :

$$Fg(\xi) = \frac{1}{T} \int_{-\infty}^{\infty} f\left(\frac{x}{T}\right) e^{-2\pi i x \xi} dx = \int_{-\infty}^{\infty} f(y) e^{-2\pi i y \cdot T\xi} dy = Ff(T\xi)$$

PSF-2  $\Rightarrow$

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) e^{-2\pi i k \xi} \leq \sum_{k=-\infty}^{\infty} Ff(T(\xi+k)) , \xi \in \mathbb{R}$$

Change  $\xi$  to  $\frac{\xi}{T}$  to conclude,

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) e^{-2\pi i k \frac{\xi}{T}} = \sum_{k=-\infty}^{\infty} Ff(\xi + Tk) , \xi \in \mathbb{R} \quad (2)$$

Replacing in (1)

$$\frac{1}{T} \left( \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) e^{-2\pi i k \frac{\xi}{T}} \right) x_{[-\frac{T}{2}, \frac{T}{2}]}(\xi) = Ff(\xi) \quad (3)$$

Take the inverse Fourier transform of both sides

$$\begin{aligned} f(x) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{T} \left( \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) e^{-2\pi i k \frac{\xi}{T}} \right) e^{2\pi i x \cdot \xi} d\xi \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{2\pi i (x - \frac{k}{T}) \xi} d\xi \end{aligned}$$

(Ex 1.1)

$$= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) \sin \frac{\omega c(Tx - k)}{\pi c(Tx - k)}$$

$$\xi \Leftrightarrow x - \frac{k}{T}$$

$\overbrace{\hspace{28em}}$   $x$   $\overbrace{\hspace{28em}}$

### 1.5. FAST FOURIER TRANSFORM (FFT)

(DFT)  $f \in S_N$ ;  $\hat{f}(k) = \sum_{n=0}^{N-1} f(n) e^{-\frac{2\pi i k n}{N}}$ ,  $k=0, \dots, N-1$

DFT requires  $\approx N^2$  operations (additions & multipl.).

FFT is an algorithm to compute DFT when  $N=2^l$ , and reduces the complexity of computations to  $\approx N \log_2 N$

$$N=2^l$$

$$\begin{aligned} \text{Even frequencies: } \hat{f}(2k) &= \sum_{n=0}^{\frac{N}{2}-1} f(n) e^{-\frac{2\pi i (2k)n}{N}} + \sum_{n=\frac{N}{2}}^{N-1} f(n) e^{-\frac{2\pi i (2k)n}{N}} \\ &= \sum_{n=0}^{\frac{N}{2}-1} f(n) e^{-\frac{2\pi i kn}{N}} + \sum_{m=0}^{\frac{N}{2}-1} f(m+\frac{N}{2}) e^{-\frac{2\pi i k(m+\frac{N}{2})}{N}} \\ &= \sum_{n=0}^{\frac{N}{2}-1} [f(n) + f(m+\frac{N}{2})] e^{-\frac{2\pi i kn}{N}} \end{aligned} \quad (6)$$

$\hat{f}(2k)$  can be computed using FFT of  $f_{\text{even}}(n) = f(n) + f(m+\frac{N}{2}) \in S_{N/2}$

Odd frequencies

$$\begin{aligned} \hat{f}(2k+1) &= \sum_{n=0}^{\frac{N}{2}-1} f(n) e^{-\frac{2\pi i (2k+1)n}{N}} + \sum_{n=\frac{N}{2}}^{N-1} f(n) e^{-\frac{2\pi i (2k+1)n}{N}} \\ &= \sum_{n=0}^{\frac{N}{2}-1} e^{-\frac{2\pi i n}{N}} f(n) e^{-\frac{2\pi i kn}{N}} + \sum_{m=0}^{\frac{N}{2}-1} e^{-\frac{2\pi i (m+\frac{N}{2})}{N}} f(m+\frac{N}{2}) e^{-\frac{2\pi i k(m+\frac{N}{2})}{N}} \\ &= \sum_{n=0}^{\frac{N}{2}-1} e^{-\frac{2\pi i n}{N}} f(n) e^{-\frac{2\pi i kn}{N}} + \sum_{m=0}^{\frac{N}{2}-1} e^{-\frac{2\pi i m}{N}} (-1) e^{-\frac{2\pi i km}{N}} f(m+\frac{N}{2}) \\ &= \sum_{n=0}^{\frac{N}{2}-1} e^{-\frac{2\pi i n}{N}} [f(n) - f(m+\frac{N}{2})] e^{-\frac{2\pi i kn}{N}} \end{aligned} \quad (7)$$

$\hat{f}(2k+1)$  can be computed using FFT for  $f_{\text{odd}}(n) = e^{-\frac{2\pi i n}{N}} [f(n) - f(n+\frac{N}{2})] \in S_{N/2}$

## 1.4 DISCRETE FOURIER TRANSFORM (DFT)

"Signals are sample and represented by numbers"

Sample rate  $T=1$ ,  $\{f(n)\}_{n \in \mathbb{Z}}$ ; In practice only  $N$  samples are taken,  $f = \{f(n)\}_{n=0}^{N-1}$  is a "finite signal of size  $N$ ". Fourier transform has to be defined in this context.

$S_N$  = discrete (complex) signals of size  $N$ ;  $S_N$  is a Hilbert space with inner product (of dim  $N$ )

$$\langle f, g \rangle = \sum_{n=0}^{N-1} f(n) \overline{g(n)} \quad (4)$$

Thm 1.2

$$\{e_n(n)\}_{n=0}^{N-1} := \left\{ \left( e^{\frac{2\pi i k n}{N}} \right)_{n=0}^{N-1} \right\}_{k=0}^{N-1} \text{ is an o.g. basis of } S_N$$

P/ See page 1.2.3 of my notes

$$\text{If } f \in S_N, f = \sum_{k=0}^{N-1} \lambda_k e_k \text{ and } \langle f, e_m \rangle = \langle \sum_{k=0}^{N-1} \lambda_k e_k, e_m \rangle$$

$$= \sum_{k=0}^{N-1} \lambda_k \langle e_k, e_m \rangle \stackrel{?}{=} \langle e_m, e_m \rangle \stackrel{?}{=} \|e_m\|^2 \stackrel{?}{=} 1/N \therefore \lambda_m = \frac{1}{N} \langle f, e_m \rangle$$

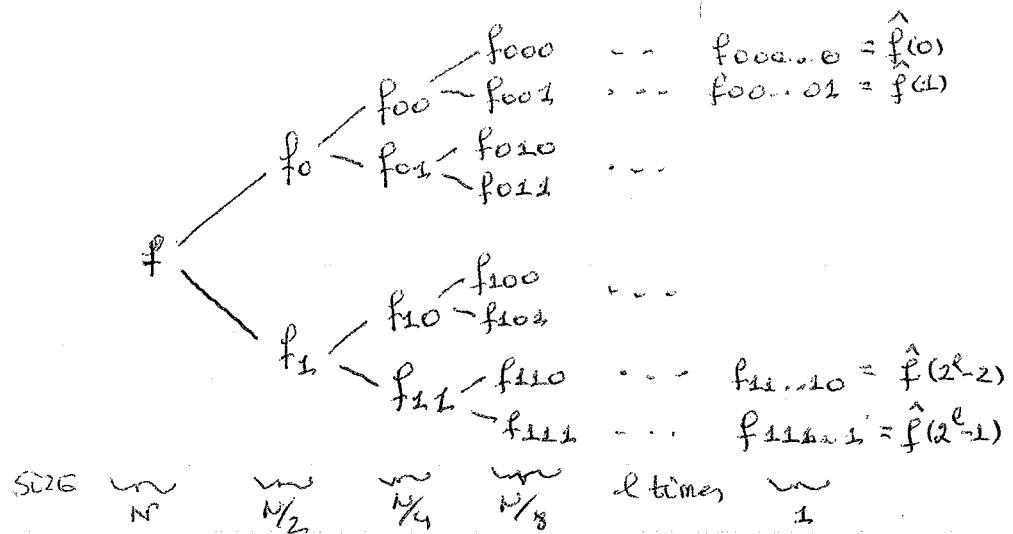
and

$$f = \frac{1}{N} \sum_{k=0}^{N-1} \langle f, e_k \rangle e_k \quad (5)$$

Notation:  $\hat{f}(k) := \langle f, e_k \rangle = \sum_{n=0}^{N-1} f(n) e^{-\frac{2\pi i k n}{N}}$  is DFT of  $f$   
 $k=0, \pm 1, \dots, N-1$

$$(5) \Rightarrow f(n) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}(k) e^{\frac{2\pi i k n}{N}} \quad (\text{Inverse DFT})$$

Ex 1.3. Plancherel: if  $f \in S_N$ ,  $\|f\|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\hat{f}(k)|^2$ .



$C(N) = \# \text{ of complex operations needed to calculate DFT using}$

(6) and (7) for  $N=2^l$

(6) (7)

$$C(N) = 2C\left(\frac{N}{2}\right) + \frac{N}{2} + \left(\frac{N}{2} + \frac{N}{2}\right) = 2C\left(\frac{N}{2}\right) + \frac{3N}{2}$$

and  $C(1)=0$  because of  $f \in S_1$ ,  $\hat{f}(0)=f(0)$

$$\begin{aligned} C(2^l) &= 2C(2^{l-1}) + \frac{3}{2} \cdot 2^l = 2[2C(2^{l-2}) + \frac{3}{2} \cdot 2^{l-1}] + \frac{3}{2} \cdot 2^l \\ &= 4C(2^{l-2}) + 2 \cdot \frac{3}{2} \cdot 2^l = 4[2C(2^{l-3}) + \frac{3}{2} \cdot 2^{l-2}] + 2 \cdot \frac{3}{2} \cdot 2^l \\ &= 2^3 C(2^{l-3}) + 3 \cdot \frac{3}{2} \cdot 2^l = \dots = \\ &= 2^l C(1) + l \cdot \frac{3}{2} \cdot 2^l = \frac{3}{2} l \cdot 2^l = \frac{3}{2} N \log_2 N \end{aligned}$$

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2.1

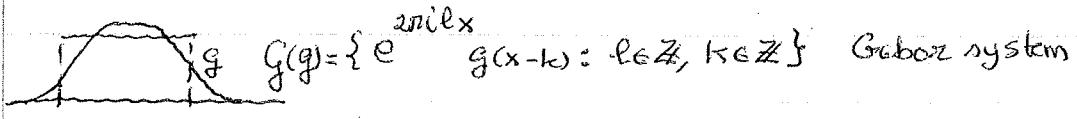
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LECTURE 2

2.1. INTRODUCTION TO WAVELETS

History of wavelets from my talk at UCM3 -2015

- $\{e^{2\pi i kx} \chi_{[k, k+1]}(x) : k \in \mathbb{Z}\}$  is an o.n.b. of  $L^2(\mathbb{R})$



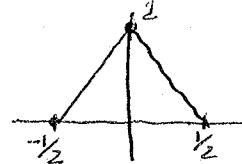
When is  $G(g)$  an o.n.b. of  $L^2(\mathbb{R})$ ?

[Balakir-low '80s]: If  $G(g)$  o.n.b. of  $L^2(\mathbb{R})$ , then either

$$\int_{-\infty}^{\infty} |x g(x)|^2 dx = \infty \quad \text{or} \quad \int_{-\infty}^{\infty} |\xi|^2 |Fg(\xi)|^2 d\xi = \infty$$

Ex 2.1. Let

$$g(x) = \begin{cases} 2x+1 & \text{if } -\frac{1}{2} \leq x < 0 \\ -2x+1 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



Show that  $G(g)$  is not an o.n. basis of  $L^2(\mathbb{R})$

Translation:  $T_y f(x) = f(x-y)$ ; Modulation:  $M_\ell g(x) = e^{2\pi i \ell x} g(x)$

i.e.  $G(g) = \{M_\ell(T_k g)(x) : \ell, k \in \mathbb{Z}\}$

Dilation:  $D_\alpha f(x) = \alpha^{-1/2} f(\alpha x)$ ,  $\alpha > 0$ .

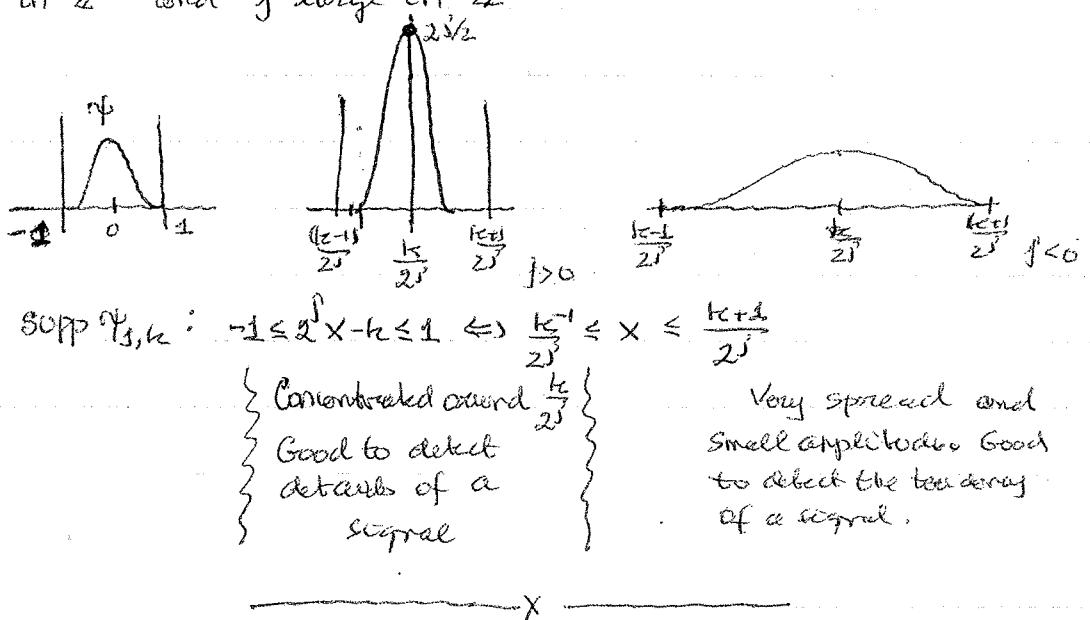
$$\|D_\alpha f\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} \alpha^{-1} |f(\alpha x)|^2 dx = \int_{-\infty}^{\infty} |f(y)|^2 dy = \|f\|_2^2$$

$\psi \in L^2(\mathbb{R})$ ,  $W(\psi) = \{D_{2^j} T_k \psi(x) : j, k \in \mathbb{Z}\}$

$$\psi_{j,k}(x) := D_{2^j} (T_k \psi)(x) = 2^{j/2} T_k \psi(2^j x) = 2^{j/2} \psi(2^j x - k)$$

Def 2.1.  $\psi \in L^2(\mathbb{R})$  is an orthonormal wavelet if  $W(\psi)$  is an o.n. basis of  $L^2(\mathbb{R})$ .

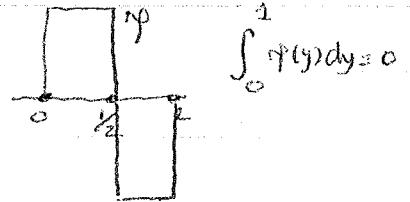
- If  $\text{Supp } \psi \subset [-1, 1]$ , draw  $\Psi_{j,k} := D_{2^j} T_{k2^j} \psi$  for  $j$  large in  $\mathbb{Z}^+$  and  $j$  large in  $\mathbb{Z}^-$



## 2.2 HAAR AND SHANNON WAVELETS

### 2.2.1 THE HAAR WAVELET

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$



$\psi$  is an o.n. wavelet in  $L^2(\mathbb{R})$ .

- O.N. System: Several steps. (See § 3.2 of my NOTES)
- BASIS?: Postpone until we know MRAs

Ex 2.2. Find the Haar coefficients  $\langle f, \Psi_{j,k} \rangle$  for all  $j, k \in \mathbb{Z}$ , when  $f = \chi_{[0, 1]}$ .

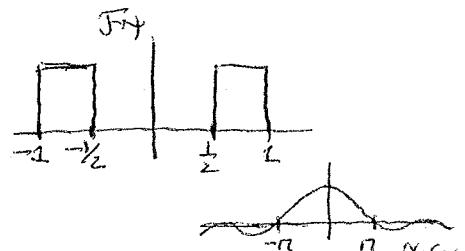
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## 2.2.2. THE SHANNON WAVELET

$$\psi \text{ s.t. } F\psi(\xi) = X_I(\xi)$$

$$I = [-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1)$$

$$(||\psi||_2 = 1)$$



$$\text{Ex 2.3. Show that } \psi(x) = \frac{\sin(2\pi x) - \sin(\pi x)}{\pi x}, \quad \psi(0) = 1$$

To show O.N need properties of  $\mathcal{F}$  and operators

Prop. 2.2  $\psi \in L^2(\mathbb{R})$ ,

$$a) \quad \mathcal{F}(T_y \psi)(\xi) = M_{-y} F\psi(\xi) \quad \forall y \in \mathbb{R}$$

$$b) \quad \mathcal{F}(M_y \psi)(\xi) = T_y F\psi(\xi) \quad \forall y \in \mathbb{R}$$

$$c) \quad \mathcal{F}(D_a \psi)(\xi) = D_{\frac{\pi}{a}} F\psi(\xi) \quad \forall a > 0$$

$$\begin{aligned} \text{P/ a) } \mathcal{F}(T_y \psi)(\xi) &= \int_{-\infty}^{\infty} T_y \psi(x) e^{-2\pi i x \xi} dx = \int_{-\infty}^{\infty} \psi(x-y) e^{-2\pi i x \xi} dx \\ &\stackrel{x-y=z}{=} \int_{-\infty}^{\infty} \psi(z) e^{-2\pi i (y+z) \xi} dz = e^{-2\pi i y \xi} \mathcal{F}\psi(\xi) = M_{-y} \mathcal{F}\psi(\xi) \end{aligned}$$

Ex 2.4. Complete the proof of Prop. 2.2.

O.N. of  $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$  ( $||\psi_{j,k}||_2 = ||\psi||_2 = ||F\psi||_2 = 1$ )

Ex 2.5. Prove

$$a) \quad \text{If } j_1, j_2 \in \mathbb{Z}, \quad j_1 \neq j_2 \quad \text{t, } k_1, k_2 \in \mathbb{Z}, \quad \langle \psi_{j_1, k_1}, \psi_{j_2, k_2} \rangle = 0$$

$$b) \quad \langle \psi_{j, k_1}, \psi_{j, k_2} \rangle = \delta_{k_1, k_2}, \quad j \in \mathbb{Z}, \quad k_1, k_2 \in \mathbb{Z}$$

- Basis enough to show that for all  $f \in C_c^2(\mathbb{R})$

$$\|f\|^2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle f, \chi_{j,k} \rangle|^2 \quad (1)$$

We need:

(A)  $\{e^{2\pi i kx} : k \in \mathbb{Z}\}$  o.n. basis of  $I = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$

(B)  $\sum_{j \in \mathbb{Z}} \chi_j(2^j \xi) = 1$  for all  $\xi \in \mathbb{R} \setminus \{0\}$

Parseval

$$\chi_{j,k} = D_{2^j} T_k \psi$$

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle f, \chi_{j,k} \rangle|^2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle Ff, T_k \rangle|^2$$

$$\stackrel{\text{Prop 2.2}}{=} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left| \int_I Ff(\xi) e^{-2\pi i k \xi} d\xi \right|^2$$

$$\eta = 2^{-j} \xi$$

$$= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left| \int_I Ff(2^j \eta) e^{-2\pi i k \eta} d\eta \right|^2$$

$$\stackrel{(A)}{=} \sum_{j \in \mathbb{Z}} 2^j \int_I |Ff(2^j \eta)|^2 d\eta = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} |\chi_j(2^j \xi)|^2 d\xi$$

$$\stackrel{(B)}{=} \int_{\mathbb{R}} |Ff(\xi)|^2 d\xi = \|f\|^2_2. \text{ Thus proves (1)}$$

~~~~~ x ~~~~~

### 2.3 MULTIRESOLUTION ANALYSIS AND PROPERTIES

Def 2.3 An MRA is a collection  $\{V_j : j \in \mathbb{Z}\}$  of closed linear subspaces of  $L^2(\mathbb{R})$  s.t.

- (1)  $V_j \subset V_{j+1} \quad \forall j \in \mathbb{Z}$
- (2)  $f \in V_j \Leftrightarrow f(2^{-j} \cdot) \in V_0$   $\text{Def } \varphi \in V_0$
- (3)  $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$
- (4)  $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$
- (5)  $\exists \varphi \in V_0$  o.t.  $\{\Psi_k \varphi : k \in \mathbb{Z}\}$  o.n. basis of  $V_0$  (Scaling)

Remark 1 (1), (2) and (5)  $\Rightarrow$  (3) [see [HW], Thm 1.6 in Chapter 2]

Remark 2. If (1), (2) and (5) hold and  $|F\varphi(z)|$  is continuous at zero,  $\therefore (4) \Leftrightarrow |F\varphi(0)| = 1$  [see Thm 1.7 in Chapter 2 of [HW]]

Prop 2.4:  $g \in L^2(\mathbb{R})$ ;  $\{g(\cdot - k) : k \in \mathbb{Z}\}$  o.n. system in  $L^2(\mathbb{R})$   
 $\Leftrightarrow \sum_{k \in \mathbb{Z}} |Fg(z + kz)|^2 = 1 \quad \text{a.e. } z \in (0, 1)$

P/ See page 3.3.3 of my Notes: compute  $\langle g, T_k g \rangle_{L^2(\mathbb{R})}$ .

Prop 2.5:  $\varphi \in L^2(\mathbb{R})$  scaling function of an MRA. Let

$$\varphi_{j,k}(x) = D_{2^j} T_k \varphi(x) = 2^{\frac{j}{2}} \varphi(2^j x - k).$$

Then  $\{\varphi_{j,k} : k \in \mathbb{Z}\}$  is an o.n. basis of  $V_j$

P/ See pages 3.3.4 and 3.3.5 of my Notes

Interpretations of Properties (3) and (4):

$P_{V_j} : L^2(\mathbb{R}) \rightarrow V_j$  orthog. projection. By Prop 2.5

$$P_{V_j} f = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{j,k} \rangle \varphi_{j,k} \quad (\text{in } L^2(\mathbb{R})) \quad (2)$$

$$(3) \Rightarrow \lim_{j \rightarrow -\infty} \|P_{V_j} f\|_2 = 0; \quad (4) \Rightarrow \lim_{j \rightarrow \infty} \|P_{V_j} f - f\|_2 = 0$$

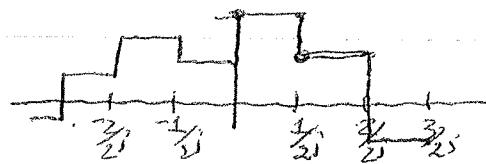
$\therefore P_{V_j} f$  is an approximation to  $f$  when  $j \rightarrow \infty$ . (Tendency)

Let  $W_j$  be s.t.  $V_j \oplus W_j = V_{j+1}$ ;  $P_{V_j} + P_{W_j} = P_{V_{j+1}}$

$\therefore P_{W_j} f = P_{V_{j+1}} f - P_{V_j} f$  encodes the difference of the image at different levels. This difference will be given with the wavelet wts to be constructed.

### Example 1 (Haar MRA)

$$V_j = \{f \in L^2(\mathbb{R}) : f \equiv \text{constant on } [\frac{k}{2^j}, \frac{k+1}{2^j}), k \in \mathbb{Z}\}$$



The scaling function is  $\varphi(x) = \chi_{[0,1)}(x)$

(1) ✓ (2) Checks (5) ✓ (3) Always true by Remark 1

To see (4):

$$F\varphi(\xi) = F(\chi_{[0,1)})(\xi) = e^{-i\mu\xi} \frac{\sin \pi \xi}{\pi \xi} \quad \text{if } F\varphi(0) = 1$$

Since  $F\varphi$  is continuous at  $\xi=0$ , (4) follows from Remark 2

- \* Write formula (2) for the Haar MRA

$$\varphi_{j,k} = 2^{j/2} \varphi(2^j x - k) = 2^{j/2} \chi_{\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)}(x)$$

$$\langle f, \varphi_{j,k} \rangle = \int_{\mathbb{R}} f(x) \overline{\varphi_{j,k}(x)} dx = \int_{\mathbb{R}/2^j} f(x) \cdot 2^{j/2} dx$$

$$P_{V_j} f = \sum_{k \in \mathbb{Z}} 2^j \left( \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} f(x) dx \right) \chi_{\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)}$$

The coefficients of this approximation are the mean value of over the intervals  $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]$ .

### EXAMPLE 2 (Shannon MRA)

Scaling function  $\varphi : F\varphi(\xi) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) \therefore \varphi(x) = \frac{\sin \pi x}{\pi x}$   
with  $\varphi(0) = 1$ .

$$\begin{aligned} V_0 &= \overline{\text{span}} \{ T_k \varphi : k \in \mathbb{Z} \}. \text{ Since } \sum_{k \in \mathbb{Z}} |F\varphi(\xi + k)|^2 = \\ &= \sum_{k \in \mathbb{Z}} \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\xi + k) = 1, \quad \xi \in \mathbb{R}, \text{ by Prop 2.4, } \{ \varphi(\cdot - k) : k \in \mathbb{Z} \} \\ &\text{o.n. basis of } V_0, \text{ which is } (S). \text{ Since } F(T_k \varphi)(\xi) = e^{-2\pi i k \xi} F\varphi(\xi) \\ &= e^{-2\pi i k \xi} \chi_{[-\frac{1}{2}, \frac{1}{2}]}, \quad V_0 = \{ f \in L^2(\mathbb{R}) : \sup Ff \subset [-\frac{1}{2}, \frac{1}{2}] \} \end{aligned}$$

Define  $V_j := \overline{\text{span}} \{ \varphi_{j,k} : k \in \mathbb{Z} \}$

$$\begin{aligned} F(\varphi_{j,k})(\xi) &= F(D_{2^j} T_k \varphi)(\xi) = D_{2^j} M_{-k} F\varphi(\xi) = 2^{-j/2} (M_{-k} F\varphi)(2^{-j} \xi) \\ &= 2^{-j/2} e^{-2\pi i 2^{-j} \xi k} F\varphi(2^{-j} \xi) = 2^{-j/2} e^{-2\pi i 2^{-j} \xi k} \cdot \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) \end{aligned}$$

$$\therefore V_j = \{ f \in L^2(\mathbb{R}) : \sup Ff \subset [-\frac{1}{2}, \frac{1}{2}] \}$$

(1) ✓, (2) ✓, (5) ✓     (3) follows by Remark 1  
Since  $F\varphi(0)=1$  &  $F\varphi(\xi)$  is zero at zero, (4) follows  
from Remark 2.

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## LECTURE 3

## 3.1. DESIGNING WAVELETS FROM AN MRA

## 3.1.1. FILTERS ASSOCIATED WITH AN MRA

$(\{V_j\}_{j \in \mathbb{Z}}, \varphi)$  is an MRA for  $L^2(\mathbb{R})$ .  $\frac{1}{2}\varphi\left(\frac{x}{2}\right) \in V_1 \subset V_0$

$$\therefore \frac{1}{2}\varphi\left(\frac{x}{2}\right) = \sum_{k=-\infty}^{\infty} h[k] \varphi(x-k) \quad (\text{in } L^2(\mathbb{R})) \quad (1)$$

with  $\text{for some } \{h[k]\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$

$$h[k] = \langle \frac{1}{2}\varphi\left(\frac{x}{2}\right), T_k \varphi \rangle = \int_{\mathbb{R}} \frac{1}{2}\varphi\left(\frac{x}{2}\right) \overline{\varphi(x-k)} dx \quad (2)$$

Take  $F$  in (1)

$$(F\varphi)(2\xi) = \left( \sum_{k=-\infty}^{\infty} h[k] e^{-2ik\xi} \right) F\varphi(\xi) \quad \xi = h(\xi) F\varphi(\xi) \quad (3)$$

and  $h(\xi)$  is called low pass filter of the MRA. (Also, the transference function) and belongs to  $L^2(0, 1)$  since  $\{h[k]\} \in \ell^2(\mathbb{Z})$

Prop 3.1. The low pass filter  $h$  of an MRA satisfies

$$|h(\xi)|^2 + |h(\xi + \frac{1}{2})|^2 = 1 \text{ a.e. } \xi \in \mathbb{R}. \text{ If } |F\varphi(0)|=1, \text{ then } |h(0)|=1$$

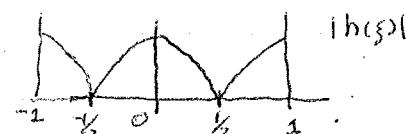
P/ See page 3.4.2 of my NOTES.

Ex 3.1. Consider Haar-MRA with scaling function  $\varphi = \chi_{[0, 1]}$

Show that  $h[0] = h\left(\frac{1}{2}\right) = \frac{1}{2}$  and  $h[k] = 0$  if  $k \neq 0, 1$ .

Deduce that

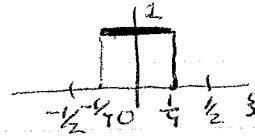
$$h(\xi) = \frac{1}{2} + \frac{1}{2} e^{-2i\xi} = e^{-i\xi} \cos \xi$$



Ex 3.2. Consider Shannon-MRA with  $\varphi(x) = \frac{\sin \pi x}{\pi x}$ ,  $\varphi(0)=1$ .

Show that  $h(\xi) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\xi)$  in  $[-\frac{1}{2}, \frac{1}{2}]$  and

$$h(k) = \begin{cases} 0 & \text{if } k=2l, l \neq 0 \\ \frac{1}{2} & \text{if } k=0 \\ (-1)^l \frac{1}{\pi(2l+1)} & \text{if } k=2l+1 \end{cases}$$



### 3.2.2. MALUAT'S RECIPE TO DESIGN WAVELETS

$(\{V_j : j \in \mathbb{Z}\}, \psi)$  MRA for  $L^2(\mathbb{R})$ . Let  $W_0$  be such that  $V_0 \oplus W_0 = V_1$ . Define  $W_j = \{g(x) = f(2^j x) : f \in W_0\}$ . It can be seen that  $V_j \oplus W_j = V_{j+1}$ . Hence,

$$V_{j+1} = V_j \oplus W_j = V_j \oplus W_{j-1} \oplus W_j = \dots = \bigoplus_{l=-\infty}^{j-1} W_l \quad (\text{by MRA-3})$$

$$\text{Since } \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}), \quad L^2(\mathbb{R}) = \bigoplus_{l=-\infty}^{j-1} W_l \quad (4)$$

Strategy: Find  $\psi \in W_0$  s.t.  $\{T_k \psi : k \in \mathbb{Z}\}$  o.n. basis of  $W_0$ . Then  $\{D_2 T_k \psi = \psi_{2k} : k \in \mathbb{Z}\}$  is an o.n. basis of  $W_1$ . By (4),  $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$  o.n. basis of  $L^2(\mathbb{R})$

#### Properties of $\psi$

(A) If  $\psi \in W_0 \Rightarrow \frac{1}{2} \psi\left(\frac{x}{2}\right) \in W_{-1} \subset V_0$

$$\therefore \frac{1}{2} \psi\left(\frac{x}{2}\right) = \sum_{k \in \mathbb{Z}} g[k] \psi(x-k) \quad (\text{in } L^2(\mathbb{R})) \quad (5)$$

with  $\{g[k]\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$  and

$$g[k] = \langle \frac{1}{2} \psi\left(\frac{x}{2}\right), \psi(x-k) \rangle = \int_{-\infty}^{\infty} \frac{1}{2} \psi\left(\frac{x}{2}\right) \overline{\psi(x-k)} dx \quad (6)$$

As in (3)

$$(\mathcal{F}\psi)(2s) = \left( \sum_{k \in \mathbb{Z}} g[k] e^{-2ik\pi s} \right) \mathcal{F}\psi(s) \Leftrightarrow g(s) \mathcal{F}\psi(s) \quad (7)$$

and  $g(s)$  is called the high pass filter of the MRA.

(B) \*\* If  $\{T_{k\varphi} : k \in \mathbb{Z}\}$  o.n. system in basis of  $W_0$ ,

$$|g(s)|^2 + |g(s + \frac{1}{2})|^2 = 1 \quad a.e. s \in \mathbb{R} \quad (8)$$

Same proof as the one in Prop 3.1.

(C) If  $W_0 = \overline{\text{span}} \{T_{k\varphi} : k \in \mathbb{Z}\} \perp V_0 = \overline{\text{span}} \{T_{k\psi} : k \in \mathbb{Z}\}$ , then

$$\sum_{k=-\infty}^{\infty} F_\psi(s+k) \overline{F_\psi(s+k)} = 0 \quad a.e. s \in \mathbb{R}. \quad (9)$$

P/ See page 3.4.6 of my NOTES

(D) If  $\{T_{k\varphi} : k \in \mathbb{Z}\}$  o.n. basis of  $W_0$ ,

$$g(s) \overline{h(s)} + g(s + \frac{1}{2}) \overline{h(s + \frac{1}{2})} = 0 \quad a.e. s \in \mathbb{R} \quad (10)$$

P/ See page 3.4.9 of my notes

Remark. If  $F_\varphi(0) = 1 \Rightarrow h(0) = 1 \Rightarrow h(\frac{1}{2}) = 0 \Rightarrow$

$$g(s) = 0 \stackrel{(2)}{\Rightarrow} F_\psi(s) = 0 \Leftrightarrow \int_{-\infty}^{\infty} \psi(x) dx = 0$$

Prop. 3.2  $\{T_{k\varphi} : k \in \mathbb{Z}\}$  o.n. basis of  $W_0 \Leftrightarrow (8) \& (10)$

Only  $\Leftarrow$  remains to be proved.

Thm. 3.3 (Mallat, 1989)

$(\{V_j : j \in \mathbb{Z}\}, \varphi)$  MRA for  $L^2(\mathbb{R})$  with low pass filter  $h$ . Define

$$g(\xi) = e^{-2\pi i \xi} \overline{h(\xi + \frac{1}{2})} \vartheta(2\xi) \quad (11)$$

for any 1-periodic function  $\vartheta$  with  $|\vartheta(\xi)|=1$  a.e.  $\xi \in \mathbb{R}$ . Then  $\psi$  is given by

$$\mathcal{F}\psi(2\xi) = g(\xi) \mathcal{F}\varphi(\xi) \quad \text{a.e. } \xi \in \mathbb{R},$$

then  $\{f_k : k \in \mathbb{Z}\}$  is an o.n. basis of  $L^2(\mathbb{R})$

P/ By Prop 3.2 & Estrategy, enough to show (8) & (10).

Since  $|h(\xi)|^2 + |h(\xi + \frac{1}{2})|^2 = 1$  a.e. (Prop 3.1), then

$$|g(\xi)|^2 + |g(\xi + \frac{1}{2})|^2 = |h(\xi + \frac{1}{2})|^2 + |h(\xi)|^2 = 1 \text{ a.e.}$$

which proves (8). To prove (10)

$$\begin{aligned} \overline{g(\xi) h(\xi)} + \overline{g(\xi + \frac{1}{2}) h(\xi + \frac{1}{2})} &= \\ e^{-2\pi i \xi} \overline{h(\xi + \frac{1}{2})} \vartheta(2\xi) \overline{h(\xi)} &+ e^{-2\pi i (\xi + \frac{1}{2})} \overline{h(\xi)} \vartheta(2\xi) \overline{h(\xi + \frac{1}{2})} \\ &= e^{-2\pi i \xi} \overline{h(\xi + \frac{1}{2})} \overline{h(\xi)} \vartheta(2\xi) [1 + e^{-\pi i}] = 0. \end{aligned}$$

X

Ex 3.3. If  $g(\xi)$  is given by (11) with  $\vartheta(\xi) = 1$ , show that  $g[k] = \overline{h[1-k]} (-1)^{1-k}$ ,  $k \in \mathbb{Z}$

X

$$\mathcal{F}\psi(2\xi) \stackrel{(7)}{=} g(\xi) \mathcal{F}\varphi(\xi) = \sum_{k \in \mathbb{Z}} g[k] \mathcal{F}\varphi(\xi) e^{-2\pi i k \xi}$$

Ex 3.3

$$\sum_{k \in \mathbb{Z}} \overline{h(1-k)} (-1)^{1-k} F(T_k \varphi)(\xi) = F\left(\sum_{k \in \mathbb{Z}} \overline{h(1-k)} (-1)^{1-k} T_k \varphi\right)(\xi)$$

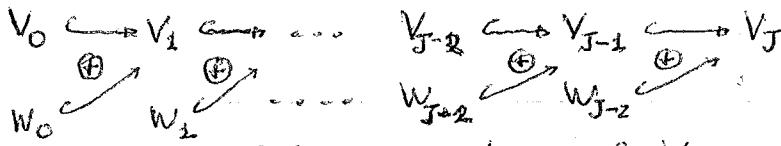
$$\therefore \frac{1}{2}\varphi\left(\frac{x}{2}\right) = \sum_{k \in \mathbb{Z}} (-1)^{1-k} \overline{h(1-k)} \varphi(x-k).$$

$$\varphi(x) = 2 \sum_{k=-\infty}^{\infty} (-1)^{1-k} \overline{h(1-k)} \varphi(2x-k) \quad (12)$$

Ex 3.4. Use (12) and Ex 3.1. to show that Mallat's recipe for Haar-MRA with  $\varphi(x) = \chi_{[0,1]}(x)$  gives

$$\varphi(x) = \underbrace{\chi_{[0,1]}(x)}_{\lambda} - \underbrace{\chi_{[1,2]}(x)}_{\lambda}$$

## 3.5. FAST WAVELET TRANSFORM



Since  $\{\varphi_{j,k} : k \in \mathbb{Z}\}$  is an o.n. basis of  $V_j$

$$P_{V_j} f = \sum_{k \in \mathbb{Z}} c_{j,k} \varphi_{j,k} \quad \text{with } c_{j,k} = \langle f, \varphi_{j,k} \rangle \quad (13)$$

(approximation)

Since  $\{\psi_{j,k} : k \in \mathbb{Z}\}$  is an o.n. basis of  $W_j$

$$P_{W_j} f = \sum_{k \in \mathbb{Z}} d_{j,k} \psi_{j,k} \quad \text{with } d_{j,k} = \langle f, \psi_{j,k} \rangle \quad (14)$$

Since  $P_{V_j} f + P_{W_j} f = P_{V_{j+1}} f$ , (14) give the details

OBJECTIVE: knowing  $c_{j,k}$  and coeff. factors  $h[k]$  and  
 $g[k] = (-1)^{k-1} h[1-k]$ , find formulas for  $c_{j-1,k}$  and  
 $d_{j-1,k}$  - Decomposition algorithm -

Lemma 3.4.  $\varphi_{j-1,p} = \sqrt{2} \sum_{k \in \mathbb{Z}} h[k-2p] \varphi_{j,k} \quad (C_0 \in L^2(\mathbb{R}))$

$$\varphi_{j-1,p} = \sqrt{2} \sum_{k \in \mathbb{Z}} g[k-2p] \psi_{j,k} \quad (C_0 \in L^2(\mathbb{R}))$$

P/ Recall

$$\frac{1}{2} \varphi\left(\frac{x}{2}\right) = \sum_{k \in \mathbb{Z}} h[k] \varphi(x-k), \quad h[k] = \langle \frac{1}{2} \varphi\left(\frac{x}{2}\right), \varphi(x-k) \rangle \quad (\#)$$

Since  $\varphi_{j-1,p} \in V_j$ ,

$$\varphi_{j-1,p} = \sum_{k \in \mathbb{Z}} \langle \varphi_{j-1,p}, \varphi_{j,k} \rangle \varphi_{j,k}$$

$$\langle \varphi_{j-1,p}, \varphi_{j,k} \rangle = \int_{-\infty}^{\infty} 2^{-j} \varphi(2^{-j}x-p) 2^{-j} \varphi(2^{-j}x-k) dx$$

$$\begin{aligned}
 y &= dx - 2p \\
 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} \varphi\left(\frac{y}{2}\right) \overline{\varphi(y+2p-k)} dy \\
 &= \sqrt{2} \left\langle \frac{1}{2} \varphi\left(\frac{y}{2}\right), \varphi(y+2p-k) \right\rangle = \sqrt{2} h[k-2p] \text{ (By 4).}
 \end{aligned}$$

Now

$$\begin{aligned}
 (13) \quad C_j(p) &= \left\langle f, \varphi_{j-1,p} \right\rangle = \underset{k \in \mathbb{Z}}{\sum} h[k-2p] C_j(k) \quad \text{Lemma 3.4}
 \end{aligned}$$

$$d_{j-1}(p) = \left\langle f, \varphi_{j-1,p} \right\rangle = \sqrt{2} \sum_{k \in \mathbb{Z}} h[k-2p] C_j(k) \quad (15)$$

which completes  $C_{j-1}(p)$  in terms of  $C_j(k)$  &  $h[k-2p]$ .

Ex 3.5 Imitate the proof of (15) to show

$$d_{j-1}(p) = \sqrt{2} \sum_{k \in \mathbb{Z}} g[k-2p] C_j(k) \quad (16)$$

with complete details  $d_{j-1}(p)$  in term of  $C_j(k)$  and  $g[k-2p]$ .

(15) & (16) are called "decomposition algorithm"

OBJECTIVE: knowing  $C_{j-1}(k)$ ,  $d_{j-1}(k)$  and the filters coeff, find a formula for  $C_j(p)$  - Reconstruction algorithm.

Since  $V_j = V_{j-1} \oplus W_{j-1}$ , the collection

$$\{ \varphi_{j-1,k} : k \in \mathbb{Z} \} \cup \{ \psi_{j-1,k} : k \in \mathbb{Z} \}$$

is an o.n. basis of  $V_j$ . Since  $\varphi_{j,p} \in V_j$ ,

$$\Psi_{j,p} = \sum_{k \in \mathbb{Z}} \langle \Psi_{j,p}, \Psi_{j,q,k} \rangle \Psi_{j-1,k} + \sum_{k \in \mathbb{Z}} \langle \Psi_{j,p}, \Psi_{j-1,k} \rangle \Psi_{j-2,k}$$

Lemma 3.4  $\sqrt{2} \sum_{k \in \mathbb{Z}} \overline{h(p-2k)} \Psi_{j-1,k} + \sqrt{2} \sum_{k \in \mathbb{Z}} \overline{g(p-2k)} \Psi_{j-2,k}$

Hence,

$$(13) \quad C_j(p) = \langle f_s \Psi_{j,p} \rangle =$$

$$= \sqrt{2} \sum_{k \in \mathbb{Z}} h(p-2k) C_{j-1}(k) + \sqrt{2} \sum_{k \in \mathbb{Z}} g(p-2k) D_{j-2}(k) \quad (17)$$

(17) is the reconstruction algorithm

————— x —————

(15) and (16) are linear equations : They can be written in matrix form :

$$\text{Let } C_j = (C_j(k))_{k \in \mathbb{Z}}^T \text{ and } D_j = (D_j(k))_{k \in \mathbb{Z}}^T$$

$$(15) \Leftrightarrow C_{j-1} = H C_j ; \quad (16) \Leftrightarrow D_{j-1} = G C_{j-1}$$

where  $H$  is a double infinite matrix  $H = (h_{jk} = \sqrt{2} h(j-2k))_{j,k \in \mathbb{Z}}$ ,  
that is all the rows of  $H$  are equal and displaced  
two units to the right. Similarly for  $G$

Abusing notation

$$(15) \& (16) \Leftrightarrow \begin{pmatrix} C_{j-1} \\ D_{j-1} \end{pmatrix} = \begin{pmatrix} H \\ G \end{pmatrix} C_j$$

————— x —————

(17) is also linear: can be written in matrix form

$$(17) \Leftrightarrow C_j = \begin{pmatrix} H^T & | & G^T \end{pmatrix} \begin{pmatrix} C_{j-1} \\ D_{j-1} \end{pmatrix}$$

Ex 3.6. Assume only  $G(0), G(1), G(2), G(3)$  are non-zero  
and consider the Haar wavelet filters  $h[0] = \frac{1}{2} = h[1]$   
and  $g[0] = -\frac{1}{2}, g[1] = \frac{1}{2}$ . Write equations (15), (16)  
and (17) in matrix form.

S/ (15)  $C_{j-1}(0) = \sqrt{2} \left[ \frac{1}{2} C_j(0) + \frac{1}{2} C_j(1) \right]; C_{j-1}(1) = \sqrt{2} \left[ \frac{1}{2} G(2) + \frac{1}{2} G(3) \right]$   
 $C_{j-1}(p) = 0 \text{ if } p \neq 0, 1.$

$$\begin{pmatrix} C_{j-1}(0) \\ C_{j-1}(1) \end{pmatrix} = \sqrt{2} \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}}_{H_{2 \times 4}} * \begin{pmatrix} C_j(0) \\ C_j(1) \\ G(2) \\ G(3) \end{pmatrix}$$

(16)

$$\begin{pmatrix} d_{j-1}(0) \\ d_{j-1}(1) \end{pmatrix} = \sqrt{2} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} G(0) \\ G(1) \\ G(2) \\ G(3) \end{pmatrix}$$

(17)  $C_j(0) = \sqrt{2} \frac{1}{2} C_{j-1}(0) + \sqrt{2} \frac{1}{2} d_{j-1}(0); G(1) = \sqrt{2} \frac{1}{2} C_{j-1}(0) + \sqrt{2} \frac{1}{2} d_{j-1}(0)$

$$C_j(2) = \sqrt{2} \frac{1}{2} G(1) + \sqrt{2} \frac{1}{2} d_{j-1}(1); G(3) = \sqrt{2} \frac{1}{2} G(1) + \sqrt{2} \frac{1}{2} d_{j-1}(1)$$

$$\begin{pmatrix} C_{j-1}(0) \\ C_{j-1}(1) \\ G(1) \\ G(3) \end{pmatrix} = \sqrt{2} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} C_{j-1}(0) \\ C_{j-1}(1) \\ d_{j-1}(0) \\ d_{j-1}(1) \end{pmatrix}$$

When sampling a signal we take  $N = 2^q$  data.

Assume  $(c_j(k))_{k=0}^{N-1}$  are the ~~data~~<sup>non-zero</sup> of data of  $c_j$  ( $j \leq q$ )

Do the decomposition algorithm  $J$  times to obtain

$c_0(k)$ ,  $0 \leq k \leq 2^{q-J}$  and  $d_j(k)$ ,  $0 \leq j \leq J$ ,  $0 \leq k \leq 2^{q-J-j}$

Discarding (small) details, a compressed signal is obtained.

~~~~~ X ~~~~

Assume  $c_j$  has only  $N = 2^q$  non zero elements,

$c_j(k)$ ,  $0 \leq k \leq N$  and  $h \sim (h[0], \dots, h[R])$  has only

the first  $R$  welf. non-zero. The number of operations  
to compute  $G_{j-1}(p)$ ,  $0 \leq p \leq \frac{N}{2}$ , is (see (15))  $\approx 2R$

To compute all of them, we need  $\approx 2k \frac{N}{2} = kN$  operations

To compute all  $d_{j-1}(p)$ ,  $0 \leq p \leq \frac{N}{2}$ , need  $\approx kN$  oper.

To compute  $G_{j-1}(p) + d_{j-1}(p)$ , need  $\approx 2kN$  oper.

Thus, after  $J$  steps, we have done  $\approx$  operations

$$2kN + 2k \frac{N}{2} + 2k \frac{N}{4} + \dots + 2k \frac{N}{2^J} \leq 2kN \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \dots \right)$$

$$\leq 4kN$$

Thus, the wavelet decomposition is an algorithm faster than FFT (this requires  $CN \log_2 N$  operations)

~~~~~ X ~~~~

LECTURE 4

## 4.1. PROPERTIES OF WAVELETS

Important for compressing: to have  $d_j(k) = \langle f, \psi_{j,k} \rangle$  small.

The number of zero moments of a wavelet  $\psi$  with compact support makes the details small

Prop 4.1.  $\psi$  o.n. wavelet in  $L^2(\mathbb{R})$  with  $\text{supp } \psi \subset [0, M]$

and

$$\int_{\mathbb{R}} \psi(x) dx = \int_{\mathbb{R}} x \psi(x) dx = \dots = \int_{\mathbb{R}} x^{p-\frac{1}{2}} \psi(x) dx = 0.$$

If  $f \in C^s$ ,  $s \geq p$ , in an open nbdg of  $\left[\frac{k_0}{2^j}, \frac{k_0+M}{2^j}\right]$ , then

$$|d_j(k)| = |\langle f, \psi_{j,k} \rangle| \leq c \lVert f \rVert_{C^s} 2^{-j(p+\frac{1}{2})}$$

P/ See page 3.6.1 of my notes.

$\int_{\mathbb{R}} \psi(x) dx = 0 \Leftrightarrow F\psi(0) = 0$ ; since  $F\psi(2s) = g(s)F\psi(s)$   
we deduce  $g(0) = 0$ ; since  $g(s) = e^{-2\pi i s} h(s + \frac{1}{2})$ , we  
deduce  $h(\frac{1}{2}) = 0$ .

Prop 4.2. If  $\psi$  is an o.n. wavelet with  $\int_{\mathbb{R}} \psi(x) dx = 0$ , then  $g'(0) = 0$  and  $h'(\frac{1}{2}) = 0$

$$P/ \frac{dF\psi}{ds}(0) = \int_{\mathbb{R}} (-2\pi i x) \psi(x) e^{-2\pi i x \cdot 0} dx \Rightarrow \frac{dF\psi}{ds}(0) = 0$$

$$\Leftrightarrow \int_{\mathbb{R}} x \psi(x) dx = 0. \text{ From } F\psi(2s) = g(s)F\psi(s) \Rightarrow$$

$$2 \frac{dF_F}{ds}(2s) = g'(s) F\varphi(s) + g(s) \frac{dF\varphi}{ds}(s)$$

$$(s=0) \Rightarrow 0 = g'(0) \cdot 1 + g(0) \cdot \frac{dF\varphi}{ds}(0) = g'(0) \cdot 1 + 0$$

$\therefore g'(0) = 0$  (We are assuming  $F\varphi(0) = 1$ ).

From  $g(s) = e^{-2\pi i s} h(s + \frac{1}{2})$ ,

$$g'(s) = -2\pi i e^{-2\pi i s} h(s + \frac{1}{2}) + e^{-2\pi i s} h'(s + \frac{1}{2})$$

$$(s=0) \Rightarrow 0 = g'(0) = -2\pi i h(\frac{1}{2}) + h'(\frac{1}{2}) = 0 + h'(\frac{1}{2})$$

$$\therefore h'(\frac{1}{2}) = 0.$$

Remark. Iterating derivatives in the proof of

Prop 4.2. It can be proved that if  $\int_{-\infty}^{\infty} x^l \varphi(x) dx = 0$ ,  $l=0, -1, p-1$ , then  $h^{(p-2)}(\frac{1}{2}) = 0$ . This has the effect of making  $h(s)$  more flat at  $\frac{1}{2}$ .

To be able to program exactly (15), (16) and (17) (wavelet decomposition and reconstruction) need  $\{h[k]\}$  and also  $\{g[k]\}$  to have only a finite number of non-zero terms. These are called finite filters. If  $h[0] \neq 0, \dots, h[N-1] \neq 0$  and  $h[k] = 0$  if  $k < 0$  and  $k \geq N$ , we say that  $\text{supp } h \subseteq [0, N]$ . In this case  $h(s) = \sum_{k=0}^{N-1} h[k] e^{-2\pi i ks}$ .

Compactly supported scaling functions  $\psi$  give rise to filters of finite length and compactly supported wavelet

Prop 4.3. Suppose  $\text{Supp } \psi \subset [0, N]$ . Then

i)  $h(g) = \sum_{-N+1 \leq k \leq N-1} h[k] e^{-2\pi i k g}$  (finite filter of length  $N+1$ )

ii)  $\text{Supp } \psi \subset [-\frac{N+1}{2}, \frac{N+1}{2}]$  ( $\psi$  has compact support)

P/ i) We know  $h[k] = \frac{1}{2} \int_{-\infty}^{\infty} \psi(\frac{x}{2}) \overline{\psi(x-k)} dx$

$\text{Supp } \psi(\frac{x}{2}) \subset [0, 2N]$  and  $\text{Supp } \psi(x-k) \subset [k, k+N]$ .

For the support to intersect;  $k+N \geq 0$  &  $k \leq 2N$ . Thus  $h[k] \neq 0$  if  $-N \leq k \leq 2N$ .

ii) Recall  $\psi(x) = 2 \sum_{k=-\infty}^{\infty} (-1)^{\frac{1-k}{2}} h[1-k] \psi(2x-k)$

$h[1-k] \neq 0 \Leftrightarrow -N+1 \leq 1-k \leq 2N-1 \Leftrightarrow 2N+2 \leq k \leq N$

$\text{Supp } \psi(2x-k) \subset [\frac{k}{2}, \frac{k+N}{2}] \therefore \frac{k}{2} \leq x \leq \frac{k+N}{2}$

Therefore,  $\text{Supp } \psi \subset [-\frac{2N+2}{2}, \frac{2N}{2}] = [-N+1, N]$

$\underline{\hspace{10cm}} x \underline{\hspace{10cm}}$

## 4.2. PROPERTIES OF FILTER COEFFICIENTS

Recall in matrix form the decomposition algorithm, with a little different notation. Start with a vector column  $\tilde{x}$  of size  $N = 2^q$ ,  $\tilde{x} = (x_0, x_1, \dots, x_{N-1})^T$ .

$$\begin{pmatrix} a \\ d \end{pmatrix} = \sqrt{2} \begin{pmatrix} H_{N/2} \\ G_{N/2} \end{pmatrix} \tilde{x} \quad (1)$$

Wavelet transf.  
Decomp. algorithm

$\tilde{a}$  = approximation (size =  $N/2$ ) ;  $d$  = details (size =  $N/2$ )

Reconstruction or inverse wavelet transform

$$\underline{x} = \sqrt{2} \begin{pmatrix} H_{N/2}^T & | & G_{N/2}^T \end{pmatrix} \begin{pmatrix} a \\ d \end{pmatrix} \quad (2)$$

$\underline{x}$

The matrix  $H$  is formed with the low pass filter coefficients.  $G$  is formed with the high pass filter coefficients. If the scaling function  $\varphi$  is known

$$h[k] = \int_{-\infty}^{+\infty} \frac{1}{2} \varphi\left(\frac{x}{2}\right) \overline{\varphi(x-k)} dx ; \quad g[k] = (-1)^{k+1} h[1-k]$$

But it is not necessary (for finite filters) to know  $\varphi$  to be able to compute  $h[k]$ . It can be done using the properties of the transference function

$$h(\zeta) = \sum_{k \in \mathbb{Z}} h[k] e^{-2\pi i k \zeta}$$

$\underline{x}$

Properties of  $h(\xi)$ :

$$(1) |h(\xi)|^2 + |h(\xi + \frac{1}{2})|^2 = 1 \quad (\text{Prop 3.1})$$

$$(2) h(0) = 1$$

$$(3) h(\frac{1}{2}) = 0$$

} low pass filter

These three properties are not independent. In fact,

(3) follows from (1) & (2) / or (2) follows from (1) & (3).

We shall use (1) & (3).

$$(3) \Leftrightarrow 0 = h(\frac{1}{2}) = \sum_{k=-\infty}^{\infty} (-1)^k h[k] \quad (4)$$

$$(2) \Leftrightarrow \sum_{k=-\infty}^{\infty} h(k) = 1 \quad (5)$$

Prop 4.4 Property (1) for  $h(\xi)$  is equivalent to

$$i) \sum_{k \in \mathbb{Z}} |h[k]|^2 = \frac{1}{2}$$

$$ii) \sum_{k \in \mathbb{Z}} h[k] \overline{h[k+2n]} = 0 \quad \forall n \in \mathbb{Z}^*$$

P/

$$1 = h(\xi) \overline{h(\xi)} + h(\xi + \frac{1}{2}) \overline{h(\xi + \frac{1}{2})} =$$

$$= \left( \sum_{k=-\infty}^{\infty} h[k] e^{-2\pi i k \xi} \right) \left( \sum_{l=-\infty}^{\infty} h[l] e^{+2\pi i l \xi} \right) +$$

$$+ \left( \sum_{k=-\infty}^{\infty} h[k] e^{-2\pi i k (\xi + \frac{1}{2})} \right) \left( \sum_{l=-\infty}^{\infty} h[l] e^{+2\pi i l (\xi + \frac{1}{2})} \right)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (h[k] \overline{h[l]} + (-1)^{k-l} h[k] \overline{h[l]}) e^{2\pi i \xi (l-k)} \quad l-k=2n$$

$$= \sum_{k=-\infty}^{\infty} \left( \sum_{l=-\infty}^{\infty} 2 h[k] \overline{h[l+2n]} \right) e^{2\pi i \xi (2n)}$$

Result follows  
by the equality  
of coefficients

Ex. 4.1 Suppose  $h \sim [h_0, h_1]$ ,  $h[1] =$  <sup>real</sup> only two non zero coefficients, and is the low pass filter of an o.n. wavelet.

Substitute the above equations for this case can show that

$$h[0] = \frac{1}{2} = h[1] \quad \text{and} \quad g[0] = -\frac{1}{2}, \quad g[1] = \frac{1}{2}$$

$$\left. \begin{array}{l} |h[0]|^2 + |h[1]|^2 = 1/2 \\ h[0] + h[1] = 1 \\ h[0] - h[1] = 0 \end{array} \right\} \Rightarrow h[0] = \frac{1}{2} = h[1]$$

$$g[k] = (-1)^{1-k} h[1-k] \Rightarrow g[0] = -\frac{1}{2}, \quad g[1] = \frac{1}{2}$$

### 4.3 DAUBECHIE'S FILTER WAVELET OF ORDER 4

Has 4 non-zero coeff:  $h(\xi) = h[0] + h[1]e^{-2\pi i \xi} + h[2]e^{-4\pi i \xi} + h[3]e^{-6\pi i \xi}$ . The conditions of section 4.1 give: (real)

$$(6) \quad h[0] + h[1] + h[2] + h[3] = 1 \quad (2)$$

$$(7) \quad h[0] - h[1] + h[2] - h[3] = 0 \quad (3)$$

$$(8) \quad h[0]^2 + h[1]^2 + h[2]^2 + h[3]^2 = \frac{1}{2} \quad \left. \begin{array}{l} (1) \\ \text{Prop 4.1} \end{array} \right\}$$

$$(9) \quad h[0]h[2] + h[1]h[3] = 0 \quad \left. \begin{array}{l} (2) \\ \text{Prop 4.1} \end{array} \right\}$$

Eq. (6) follows from (7), (8) and (9) because (1) & (3)  $\Rightarrow$  (2). We have 3 equations and 4 unknowns. Try to solve it.

$$(9) \Rightarrow (h_2, h_3) \perp (h_0, h_1) \Rightarrow (h_2, h_3) = c(-h_1, h_0)$$

$$\boxed{h_2 = -ch_1}, \boxed{h_3 = ch_0} \quad (10) \quad c \neq 0$$

$$(8) \Rightarrow h_0^2 + h_1^2 + c^2 h_1^2 + c^2 h_0^2 = \frac{1}{2}$$

$$\boxed{h_0^2 + h_1^2 = \frac{1}{2(1+c^2)}} \quad (11)$$

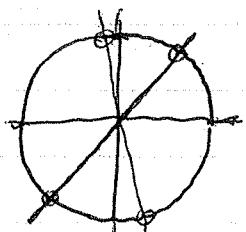
circumference of radius

$$\sqrt{\frac{1}{2(1+c^2)}}$$

$$(7) \Rightarrow h_0 - h_1 + ch_1 - ch_0 = 0$$

$$\boxed{h_1 = \frac{1-c}{1+c} h_0} \quad c \neq -1 \quad (12)$$

straight line



Infinitely many solutions

Daubechie's propose an additional condition

$$h'(1/2) = 0 \quad (\text{flatter filter at } \xi = 1/2) \quad (\text{See Prop 4.2})$$

$$h(\xi) = \sum_{k \in \mathbb{Z}} h_k e^{-2\pi i k \xi} \Rightarrow h'(\xi) = -2\pi i \sum_{k \in \mathbb{Z}} k h_k e^{-2\pi i k \xi}$$

$$\Rightarrow h'(1/2) = -2\pi i \sum_{k \in \mathbb{Z}} k h_k (-1)^k$$

$$(13) \quad \sum_{k \in \mathbb{Z}} k h_k (-1)^k = 0$$

For our filter  $(h_0, h_1, h_2, h_3)$

$$-h_1 + 2h_2 - 3h_3 = 0. \quad (14)$$

$$\text{From (10), } -h_1 - 2ch_1 - 3ch_0 = 0$$

$$h_1 = \frac{-3c}{1+2c} h_0$$

$$(15) \quad c \neq -\frac{1}{2}$$

straight line

To have solution, slopes of straight lines (13) and (15) must be equal

$$\frac{1-c}{1+c} = -\frac{3c}{1+2c} \Rightarrow c = -2 \pm \sqrt{3}$$

$$\boxed{\text{Take } c = -2 + \sqrt{3}}$$

$$(12) \Rightarrow h_1 = \frac{1-c}{1+c} h_0 = \frac{3-\sqrt{3}}{1+\sqrt{3}} h_0 = \frac{(3-\sqrt{3})(3+\sqrt{3})}{(-1+\sqrt{3})(-1-\sqrt{3})} h_0 \\ = \frac{-2\sqrt{3}}{-2} h_0 = \sqrt{3} h_0 // \quad (16)$$

$$(11) \Rightarrow 4h_0^2 = \frac{1}{2(1+c^2)} = \frac{1}{2} \cdot \frac{1}{1+(-2+\sqrt{3})^2} = \frac{2+\sqrt{3}}{8}$$

$$h_0^2 = \frac{2+\sqrt{3}}{32}$$

Before taking square roots,  $(1+\sqrt{3})^2 = 4+2\sqrt{3}=2(2+\sqrt{3})$ .

Hence

$$h_0^2 = \frac{(1+\sqrt{3})^2}{64} \Rightarrow h_0 = \pm \frac{1+\sqrt{3}}{8}$$

Take, for example  $h_0 = \frac{1+\sqrt{3}}{8}$ . Then,

$$(10) \quad h_1 = \frac{1-c}{1+c} \cdot h_0 = \sqrt{3} h_0 = \frac{3+\sqrt{3}}{8}$$

$$(10) \quad h_2 = -ch_1 = \frac{3-\sqrt{3}}{8}, \quad (10) \quad h_3 = ch_0 = \frac{1-\sqrt{3}}{8}$$

Daubechi's filter of order 4:

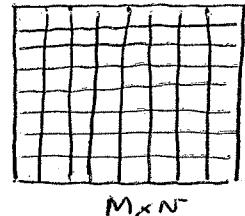
$$h_0 = \frac{1+\sqrt{3}}{8}, \quad h_1 = \frac{3+\sqrt{3}}{8}, \quad h_2 = \frac{3-\sqrt{3}}{8}, \quad h_3 = \frac{1-\sqrt{3}}{8}$$

Since  $g_k = (-1)^{1-k} h_{1-k}$

$$g_0 = -\frac{3+\sqrt{3}}{8}, \quad g_1 = \frac{1+\sqrt{3}}{8}, \quad g_{-1} = \frac{3-\sqrt{3}}{8}, \quad g_{-2} = -\frac{1-\sqrt{3}}{8}$$

#### 4.4. IMAGE PROCESSING WITH WAVELETS

A gray scale digital image can be viewed as an  $M \times N$  matrix whose entries are integer numbers between 0 and  $255 = 2^8 - 1$  ( $0 = \text{black}$ ,  $255 = \text{white}$ ).



The number indicates the gray intensity of the pixel

(Show example in transp. 2.4 of Talk UC3-M)

Let  $X = (x_{i,j})_{i=1, j=1}^{M \times N}$  a gray scale digital image.

To process  $X$  with o.n. wavelets, apply wavelet trans. to the columns of  $X$  to obtain  $N = 2^q$ ,  $M = 2^l$

$$\sqrt{2} \begin{bmatrix} H_{M/2} \\ G_{M/2} \end{bmatrix} X \quad \boxed{X} \rightarrow \begin{array}{c} \text{Aprox} \\ \text{Details} \end{array}$$

Apply now the wavelet transform to the ~~columns~~ rows of this new image to obtain

$$\sqrt{2} \begin{bmatrix} H_{M/2} \\ G_{M/2} \end{bmatrix} X \sqrt{2} \begin{bmatrix} H_{N/2}^T & G_{N/2}^T \end{bmatrix} \quad \left\{ \begin{array}{c} \boxed{\phantom{A}} \\ \boxed{\phantom{D}} \end{array} \right\} \rightarrow \begin{array}{c} A \\ V \\ H \\ D \end{array}$$

$$= 2 \begin{bmatrix} H_{M/2} X \\ G_{M/2} X \end{bmatrix} \begin{bmatrix} H_{N/2}^T & G_{N/2}^T \end{bmatrix} = 2 \begin{bmatrix} H_{M/2} X H_{N/2}^T & H_{M/2} X G_{N/2}^T \\ G_{M/2} X H_{N/2}^T & G_{M/2} X G_{N/2}^T \end{bmatrix}$$

$$:= \begin{bmatrix} A & V \\ H & D \end{bmatrix}$$

(Show transparency 8.1 of Notes UC3M-Talks)

Ex. 4.2 Compute the approximation and the details (Horizontal, vertical, and diagonal) of the image

$X = (x_{ij})_{i,j=0}^{3,3}$ ,  $a_{i,j} = 20$  if  $j=0, \dots, 3$ ;  $a_{i,i} = 20$  if  $i=0, \dots, 3$  and the rest of the coefficients zero, using a 2D-Haar transform (Use grey colors with Black=20, White=0, to have an image representation)

$$H_4^T \begin{pmatrix} 0 & 0 \\ 0 & 0 & 10 & 0 \\ 10 & 10 & 15 & 10 \\ 0 & 0 & 10 & 0 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \\ 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 \\ 10 \\ 10 \end{pmatrix}$$

LECTURE 5 : THE JPEG FORMAT FOR IMAGES

JPEG (Joint Photographic Expert Group)

Brief history of JPEG80 and JPEG2000 = JP62tc

Use transparencies Section 2.4 - JPEG (Class Notes)

Step 1. The RGB system for colors

|             |       |                 |                         |
|-------------|-------|-----------------|-------------------------|
| (255, 0, 0) | Red   | (255, 255, 0)   | Red and Green<br>Yellow |
| (0, 255, 0) | Green | (0, 255, 255)   | Green and Blue<br>Cyan  |
| (0, 0, 255) | Blue  | (255, 0, 255)   | Red and Blue<br>Magenta |
| (0, 0, 0)   | Black | (255, 255, 255) | White                   |

3-D representation of RGB system

Step 2 Color space transformation

$$\begin{pmatrix} Y \\ C_b \\ C_r \end{pmatrix} = \begin{pmatrix} 0.257 & 0.504 & 0.098 \\ -0.148 & -0.291 & 0.439 \\ 0.439 & -0.368 & -0.071 \end{pmatrix} \begin{pmatrix} R \\ G \\ B \end{pmatrix} + \begin{pmatrix} 16 \\ 128 \\ 128 \end{pmatrix}$$

Other transformation may be applied

Step 3. Downsampling

Downsampling by 2 is done eliminating every second number in the components of the color space representation.

Step 4 { 4. 1D Discrete Cosine Transform on JPEG80  
 Example of a grey scale representation of color  
 of an  $8 \times 8$  block of an image.

4.2. CDF - Biorthogonal wavelets (Coifman, Daubechies, Feauveau). Similar to 2D-orthogonal wavelet transform

$$X_{8 \times 8} \xrightarrow{\text{Transform}} \hat{X}_{8 \times 8} = (\hat{x}_{ij})_{i,j=1}^8$$

### Step 5. Quantization

$$\begin{matrix} & \begin{matrix} 16 & 11 & 10 & 16 & 24 & 40 & 52 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 24 & 13 & 16 & 24 & & & 56 \\ 18 & & & & & 62 \\ 24 & & & & & 77 \\ 49 & & & & & 92 \\ & 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{matrix} \\ Q \end{matrix} = (q_{ij})_{i,j=1}^8$$

$$B = (b_{ij})_{i,j=1}^8, b_{ij} = \left\lceil \frac{\hat{x}_{ij}}{q_{ij}} \right\rceil \text{ (rounded)}$$

Show B for the play/toy example

### Step 6. Encoding

Arrange numbers in a zig-zag order

Use Huffman encoding to compress: assign a code with fewer bits to symbols that appear more frequently. No information is lost in this step

\_\_\_\_\_ x \_\_\_\_\_

### 5.1. DISCRETE COSINE TRANSFORM

$\mathbb{S}_N$  = space of discrete signals of size  $N$  :  $(f(n))_{n=0}^{N-1}$

In Thm 1.2 (lecture 1) we show:

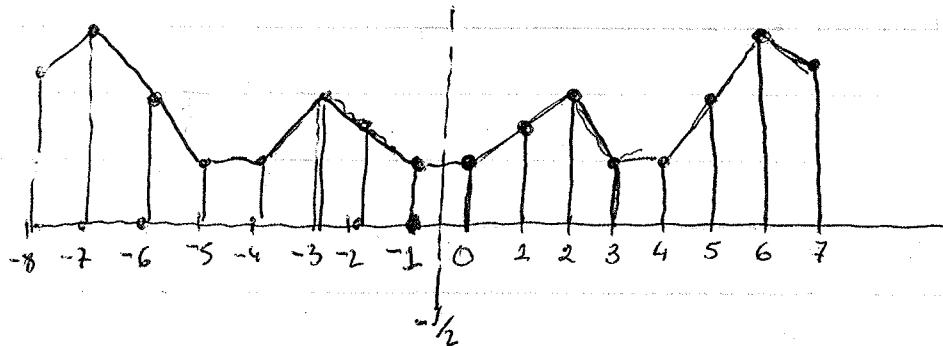
$$\left\{ e_k^{(N)} f(n) \right\}_{k=0}^{N-1} = \left\{ \frac{1}{\sqrt{N}} \left( e^{\frac{2\pi i k n}{N}} \right)_{n=0}^{N-1} \right\}_{k=0}^{N-1} \quad (1)$$

is an o.n. basis of  $\mathbb{S}_N \approx \mathbb{C}^N$

(Section 2.2.4 of my NOTES)

$(f(n))_{n=1}^N$  is extended by symmetry with respect to  $-\frac{1}{2}$  to obtain a signal of size  $2N$  given by

$$\tilde{f}(n) = \begin{cases} f(n) & \text{if } 0 \leq n \leq N-1 \\ f(-n-1) & \text{if } -N \leq n \leq -1 \end{cases}$$



In  $\mathbb{S}_{2N}$  the collection

$$\left\{ e_k^{(2N)} f(n) \right\}_{k=-N}^{N-1} = \left\{ \frac{1}{\sqrt{2N}} \left( e^{\frac{\pi i k (n+\frac{1}{2})}{N}} \right)_{n=-N}^{N-1} \right\}_{k=-N}^{N-1} \quad (2)$$

is an o.n. basis.

Ex 5.1 Show that  $\mathbb{S}_{2N}$  (2) is an o.n. basis of  $\mathbb{S}_{2N} \approx \mathbb{C}^{2N}$  using the sum of an arithmetic prog.

Thus, any  $g \in \mathbb{S}_{2N}$ , can be written as

$$g(n) = \sum_{k=-N}^N 2_{k,c} \alpha_k^{(2N)}(n), \quad n = -N, \dots, N-1$$

Use  $e^{i\theta} = \cos \theta + i \sin \theta$  to show that each  $g \in \mathbb{S}_{2N}$  can be written as linear combination of

$$\left\{ (C_k^{(2N)}(n))_{n=-N}^{N-1} \right\} \stackrel{\approx}{=} \left\{ (\cos \frac{kn}{N}(n+\frac{1}{2}))_{n=-N}^{N-1} \right\}_{k=-N}^{N-1} \quad (3)$$

$$\cup \left\{ (S_k^{(2N)}(n))_{n=-N}^{N-1} \right\} \stackrel{\approx}{=} \left\{ (\sin \frac{kn}{N}(n+\frac{1}{2}))_{n=-N}^{N-1} \right\}_{k=-N}^{N-1}$$

Collection (3) has  $4N$  elements, while  $\mathbb{S}_{2N}$  has  $2N$  elements. Thus, half of them have to be linear combination of the other half:

$$S_0^{(2N)} = (0); \quad C_0^{(2N)} = (1); \quad S_{-N}^{(2N)} = -((-1)^n)_{n=-N}^{N-1}; \quad C_N^{(2N)} = (0)$$

If  $0 < k < N$

$$(C_{-k}^{(2N)}(n))_{n=-N}^{N-1} = (C_k^{(2N)}(n))_{n=-N}^{N-1}$$

$$(S_{-k}^{(2N)}(n))_{n=-N}^{N-1} = - (S_k^{(2N)}(n))_{n=-N}^{N-1}$$

Thus, each element of  $\mathbb{S}_{2N}$  can be written as a linear combination of

$$\left\{ (C_k^{(2N)}(n))_{n=-N}^{N-1} \right\}_{k=0}^{N-1} \stackrel{\approx}{=} \left\{ (\cos \frac{kn}{N}(n+\frac{1}{2}))_{n=-N}^{N-1} \right\}_{k=0}^{N-1}$$

$$\cup \left\{ (S_k^{(2N)}(n))_{n=-N}^{N-1} \right\}_{k=1}^N \stackrel{\approx}{=} \left\{ (\sin \frac{kn}{N}(n+\frac{1}{2}))_{n=-N}^{N-1} \right\}_{k=1}^N \quad (4)$$

In terms of collection (4),  $\tilde{f} \in \mathbb{S}_{2N}$  as

$$\tilde{f}(n) = \sum_{k=0}^{N-1} a_k C_k^{(2N)}(n) + \sum_{k=1}^N b_k S_k^{(2N)}(n), \quad -N \leq n \leq N-1$$

The discrete signals  $(S_k^{(2N)}(n))_{n=-N}^{N-1}$ ,  $k=1, 2, \dots, N$   
are anti-symmetric w.r.t.  $-\frac{1}{2}$ :

$$S_k^{(2N)}(-1-n) = \sin \frac{k\pi}{N} (-1-n+\frac{1}{2}) = \sin \frac{k\pi}{N} (-n-\frac{1}{2}) = -S_k^{(2N)}(n)$$

Thus, since  $(\tilde{f}(n))_{n=-N}^{N-1}$  is symmetric w.r.t.  $-\frac{1}{2}$ ,  
we conclude  $b_k = 0$ ,  $k=1, \dots, N$ .

Since  $(f(n))_{n=0}^{N-1} = (\tilde{f}(n))_{n=0}^{N-1}$ , any  $f \in \mathbb{S}_N$   
can be written as a linear combination of  $(C_k^{(2N)}(n))_{n=0}^{N-1}$   
 $k=0, 1, \dots, N-1$

### Thm 5.1 (DC-I basis)

The collection

$$\left\{ \lambda_k \sqrt{\frac{2}{N}} (C_k^{(2N)}(n))_{n=0}^{N-1} = \lambda_k \sqrt{\frac{2}{N}} \left( \cos \frac{k\pi}{N} \left( n + \frac{1}{2} \right) \right)_{n=0}^{N-1} \right\}_{k=0}^{N-1}$$

with  $\lambda_k = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k=0 \\ 1 & \text{if } 1 \leq k \leq N-1 \end{cases}$  as an o.n. basis of  $\mathbb{S}_N$

P/ Only needed to prove orthonormality

Ex 5.2 Show that

$$\sum_{n=0}^{N-1} \cos \left( \frac{n\pi}{N} \left( n + \frac{1}{2} \right) \right) = 0, \quad k=1, 2, \dots, 2N-1 \quad (5)$$

(Use the sum of a geom. progression, §2.24 of Notes)

$$(5) \Rightarrow C_0^{(2N)} \perp C_k^{(2N)}, k=1, 2, \dots, N-1.$$

To show  $C_k^{(2N)} \perp C_\ell^{(2N)}, 1 \leq k < \ell \leq N-1$ :

$$\begin{aligned} \langle C_k^{(2N)}, C_\ell^{(2N)} \rangle &= \sum_{n=0}^{N-1} \cos \frac{k\pi}{N}(n+\frac{1}{2}) \cos \frac{\ell\pi}{N}(n+\frac{1}{2}) = \\ &= \frac{1}{2} \sum_{n=0}^{N-1} \cos \frac{(k+\ell)\pi}{N}(n+\frac{1}{2}) + \cos \frac{(k-\ell)\pi}{N}(n+\frac{1}{2}) \stackrel{(5)}{=} 0 \end{aligned}$$

Now

$$\begin{aligned} \|\lambda_0 \sqrt{\frac{2}{N}} C_0^{(2N)}\|^2 &= \frac{1}{N} \sum_{n=0}^{N-1} 1 = 1 \\ \|\lambda_1 \sqrt{\frac{2}{N}} C_1^{(2N)}\|^2 &= \frac{2}{N} \sum_{n=0}^{N-1} \cos^2 \frac{k\pi}{N}(n+\frac{1}{2}) = \\ &= \frac{1}{N} \sum_{n=0}^{N-1} 1 + \cos \frac{2k\pi}{N}(n+\frac{1}{2}) \stackrel{(5)}{=} 1. \end{aligned}$$

Using DC-I basis (Thm 5.1), for any  $f \in \mathcal{S}_N$

$$f(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} \hat{f}_I(k) \lambda_k \cos \left[ \frac{k\pi}{N}(n+\frac{1}{2}) \right], n=0, 1, \dots, N-1 \quad (6)$$

where

$$\hat{f}_I(k) = \langle f, \lambda_k \sqrt{\frac{2}{N}} C_k^{(2N)} \rangle = 2\lambda_k \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} f(n) \cos \left[ \frac{k\pi}{N}(n+\frac{1}{2}) \right] \quad (7)$$

(7) is called DCT-I transform and (6) is inverse DCT-I transform

(6) & (7) can be written in matrix form to use in MATLAB

$$(7) \Leftrightarrow \begin{pmatrix} \hat{f}_I(0) \\ \hat{f}_I(1) \\ \vdots \\ \hat{f}_I(N-1) \end{pmatrix} = \underbrace{\mathbf{F}}_{\text{FF}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \omega \frac{1}{\sqrt{2N}} & \omega \frac{3}{\sqrt{2N}} & \cdots & \omega \frac{(2N-1)}{\sqrt{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ - & - & - & - \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{pmatrix}$$

$C_I$

$$(8) \Leftrightarrow \begin{pmatrix} f(0) \\ \vdots \\ f(N-1) \end{pmatrix} = C_I^T \begin{pmatrix} \hat{f}_I(0) \\ \vdots \\ \hat{f}_I(N-1) \end{pmatrix}$$

Thus,  $\left(f(n)\right)_{n=0}^{N-1} = C_I^T (\hat{f}_I(k)) = C_I^T C_I \left(f(n)\right)_{n=0}^{N-1} \Rightarrow$   
 $C_I^T C_I = I$  i.e.  $C_I$  is an orthogonal matrix

Remark: The number of operations required to compute DCT-I coeff is  $\approx 2N^2$ . As in the case of **FF** there is a fast algorithm to allow to compute  $\hat{f}_I(k)$   $k=0, \dots, N-1$ , with  $\approx N \log_2 N$  operations

## 5.2 2D-DISCRETE COSINE TRANSFORM: FOR IMAGES

For each  $8 \times 8$  block of an image, the basis used is

$$\left( \lambda_{k,l} e^{\frac{j\pi}{8} kn} \cos \frac{k\pi}{8} (n+\frac{1}{2}) \cos \frac{j\pi}{8} lm \cos \frac{l\pi}{8} (m+\frac{1}{2}) \right)_{n,m=0}^7 \quad \text{with } 0 \leq k, l \leq N-1$$

and

$$z_p = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } p=0 \\ 0 & \text{if } 1 \leq p \leq N-1 \end{cases}$$

In matrix form, the 64 coefficients are computed with

$$\left( \hat{f}_I(k, l) \right)_{k,l=0}^7 = C_I \left( f(n, m) \right)_{n,m=0}^7 C_I^T$$

The first element of the matrix is

$$\hat{f}_I(0,0) = \frac{1}{2} \frac{1}{4} \sum_{m=0}^7 \sum_{n=0}^7 f(n, m) \quad (\text{mean value})$$

and is called the DC-coefficient of the image. It is a larger coefficient. In JPEG80, 128 is subtracted from the values of each pixel  $\{0, \dots, 255\}$  to obtain numbers between  $\{-128, \dots, 127\}$ .



### 5.3 BIORTHOGONAL FILTERS FOR JPEG2000

Daubechies proved that the only symmetric, finite length orthogonal filter is the Haar filter  $h = \left[ \frac{1}{2}, \frac{1}{2} \right]$ .

To find symmetric filters, Cohen-Daubechies-Feauveau relinquish orthogonality. They look for two finite low pass filters  $h(\xi)$  &  $\tilde{h}(\xi)$

s.t.

$$\tilde{h}(\xi) \overline{h(\xi)} + \tilde{h}(\xi + \frac{1}{2}) \overline{h(\xi + \frac{1}{2})} = 1 \quad (8)$$

$$\tilde{h}(0) = 1, \quad h(0) = 1 \quad (9)$$

$$\tilde{h}(\frac{1}{2}) = 0, \quad h(\frac{1}{2}) = 0 \quad (10)$$

Then define

$$\tilde{g}(\xi) = -e^{2\pi i \xi} \overline{\tilde{h}(\xi + \frac{1}{2})} \quad (11)$$

$$g(\xi) = -e^{2\pi i \xi} \overline{h(\xi + \frac{1}{2})} \quad (12)$$

to get high pass filters, to have

$$\tilde{g}(\xi) \overline{\tilde{g}(\xi + \frac{1}{2})} + \tilde{g}(\xi + \frac{1}{2}) \overline{g(\xi + \frac{1}{2})} = 1$$

$$\tilde{g}(0) = 0, \quad g(0) = 0$$

$$\tilde{g}(\frac{1}{2}) = 1, \quad g(\frac{1}{2}) = 1$$

————— x —————

Ex 5.3. As in Ex. 3.3, deduce from (11), (12) that

$$\tilde{g}_k = (-1)^k h_{1-k} \text{ and } g_k = (-1)^k \tilde{h}_{1-k}$$

$$\text{where } h(\xi) = \sum_{k=-\infty}^{\infty} h_k e^{2\pi i k \xi}, \quad \tilde{h}(\xi) = \sum_{k=-\infty}^{\infty} \tilde{h}_k e^{2\pi i k \xi},$$

$$g(\xi) = \sum_{k=-\infty}^{\infty} g_k e^{2\pi i k \xi}, \quad \tilde{g}(\xi) = \sum_{k=-\infty}^{\infty} \tilde{g}_k e^{2\pi i k \xi}$$

                 X                 

Clearly,

$$\tilde{h}\left(\frac{1}{2}\right) = 0 \Leftrightarrow \sum_{k=-\infty}^{\infty} (-1)^k \tilde{h}_k = 0 \quad (13)$$

$$h\left(\frac{1}{2}\right) = 0 \Leftrightarrow \sum_{k=-\infty}^{\infty} (-1)^k h_k = 0 \quad (14)$$

and

$$(8) \Leftrightarrow \sum_{k=-\infty}^{\infty} \tilde{h}_k \overline{h_{k-2n}} = \frac{1}{2} \delta_{0,n} \quad n \in \mathbb{Z} \quad (15)$$

                 X                 

The idea of CDF allows to use a simple symmetric filter for  $\tilde{h}(\xi)$ ; for example

$$\tilde{h}(\xi) = \frac{1}{4} e^{-2\pi i \xi} + \frac{2}{4} + \frac{1}{4} e^{2\pi i \xi}, \quad \tilde{h} = [1, 2, 1] \\ (\text{---} \frac{1}{2}(1+\cos 2\pi \xi))$$

(Symmetric of order 3) and find a filter  $h$  using (14) and (15).

For example, look for

$$h = (h_{+2}, h_{+1}, h_0, h_1, h_2)$$

of length 5 and symmetric. (14) and (15) give

$$(n=2) \quad \left. \begin{array}{l} h_0 - 2h_1 + 2h_2 = 0 \\ h_0 + h_1 = 1 \\ h_1 + 2h_2 = 0 \end{array} \right\}$$

$$\text{Solution: } h_0 = \frac{3}{4}, h_1 = \frac{1}{4}, h_2 = -\frac{1}{8}$$

This is called a CDF(5, 3) filter.

JPEG2000 uses CDF(9, 7) filter. The filter  
is  $\tilde{h}$  of length 7

$$\tilde{h} = \frac{1}{25} (1, 6, 15, 20, 15, 6, 1)$$

("normalize" binomial coefficients  $\binom{6}{k} \frac{1}{2^6}, k=0, \dots, 6$ )

and find  $h = (h_4, h_3, h_2, h_1, h_0, h_1, h_2, h_3, h_4)$

Symmetric of length 9 using (14), (15).

Processing digital images  $X = (x_{ij})_{i=1, j=1}^{M \times N}$ ,  $M=2^k$ ,  $N=2^l$  with biorthogonal wavelets is similar to the case of orthonormal wavelets. (See § 4.4)

First apply the wavelet transform  $W_{M/2} = \sqrt{2} \begin{bmatrix} H_{M/2} \\ G_{M/2} \end{bmatrix}$  to the columns of  $X$  to obtain

$$\sqrt{2} \begin{bmatrix} H_{M/2} \\ G_{M/2} \end{bmatrix} X$$

Then apply the wavelet transform  $\tilde{W}_{N/2} = \sqrt{2} \begin{bmatrix} \tilde{H}_{N/2} \\ \tilde{G}_{N/2} \end{bmatrix}$  (rows) to this matrix to obtain

$$\sqrt{2} \begin{bmatrix} H_{M/2} \\ G_{M/2} \end{bmatrix} \times \begin{bmatrix} \tilde{H}_{N/2}^T & \tilde{G}_{N/2}^T \end{bmatrix} \sqrt{2} := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$\overbrace{\hspace{10em}} \times \overbrace{\hspace{10em}}$

Compression is done by quantizing the coefficient (after a few iterations) and use encoding in an efficient way, such as Huffman encoding.

$\overbrace{\hspace{10em}} \times \overbrace{\hspace{10em}}$

## 5.4. ENCODING (HUFFMAN)

Suppose that after applying DCT or Biorthogonal CDF(7,9) to an  $8 \times 8$  block of an image we have obtained

|    |   |   |   |   |   |   |   |
|----|---|---|---|---|---|---|---|
| 26 | 3 | 6 | 2 | 2 | 0 | 0 | 0 |
| 0  | 2 | 4 | 1 | 1 | 0 | 0 | 0 |
| 3  | 1 | 5 | 1 | 1 | 0 | 0 | 0 |
| 4  | 1 | 2 | 1 | 0 | 0 | 0 | 0 |
| 0  | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0  | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0  | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0  | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Each number is codified with 1 byte = 8 bits

$$26 = 00011010_2 ; 3 = 00000011_2 ;$$

$$6 = 00000110_2 ; 2 = 00000010_2 ;$$

$$4 = 00000100_2 ; 5 = 00000101_2 ;$$

$$0 = 00000000_2 ; 1 = 00000001_2$$

The bit stream save in the computer is

00011010 00000011 00000000 00000011 00000010

00000110 00000010 . . . . .

. . . . . 00000000

It is not necessary to separate bytes (words) since each one has 8 bits.

Huffman found <sup>(an optimal)</sup> way to modify a collection of "words" ~~so that~~ so that the bit per pixel (bbp) count is greatly reduced.

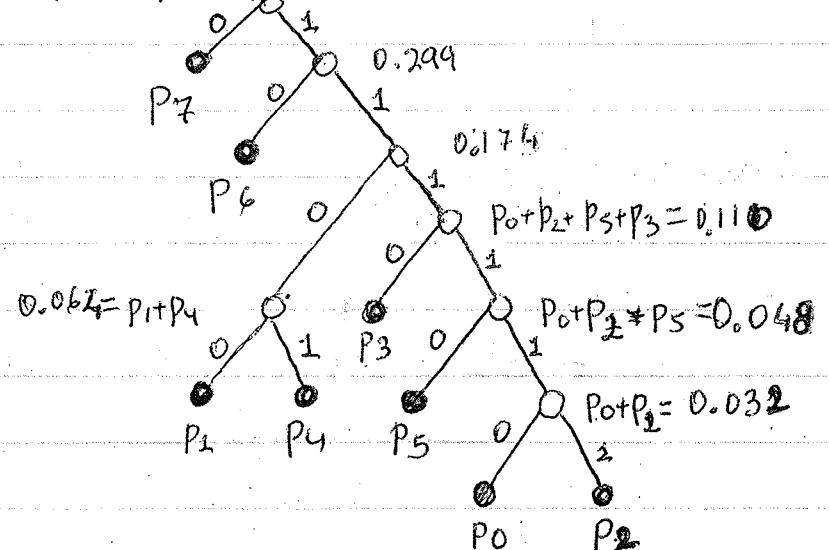
Step 1 Find the frequency of each binary digit <sup>(relative)</sup>

Digit/Word Bin. Rep. Frequency Rel. Freq.

|            |              |    |                                     |
|------------|--------------|----|-------------------------------------|
| $x_0 = 26$ | $00011010_2$ | 1  | $\frac{1}{64} = 0.0156 \approx p_0$ |
| $x_1 = 3$  |              | 2  | $\frac{2}{64} = 0.0312 \approx p_1$ |
| $x_2 = 6$  |              | 1  | $\frac{1}{64} = 0.0156 \approx p_2$ |
| $x_3 = 2$  |              | 4  | $\frac{4}{64} = 0.0625 \approx p_3$ |
| $x_4 = 5$  |              | 2  | $\frac{2}{64} = 0.0312 \approx p_4$ |
| $x_5 = 5$  |              | 1  | $\frac{1}{64} = 0.0156 \approx p_5$ |
| $x_6 = 0$  | $00000001_2$ | 8  | $\frac{8}{64} = 0.125 \approx p_6$  |
| $x_7 = 0$  | $00000000_2$ | 45 | $\frac{45}{64} = 0.703 \approx p_7$ |

Step 2 Order the relative frequencies in increasing order

$$(p_0 \leq p_2 \leq p_5 \leq p_1 \leq p_4 \leq p_3 \leq p_6 \leq p_7)$$



Step 3 Coding

$$26=x_0 = 111110 \quad 3=x_1 = 1100 \quad 6=x_2 = 111110$$

$$2=x_3 = 1110 \quad 4=x_4 = 1101 \quad 5=x_5 = 11110$$

$$1=x_6 = 10 \quad 0=x_7 = 0$$

Coded bit stream

26      3      0      3      2  
 111110 1100011001110 ...  
 . . . . .  
 38  
 . . . . 10 0 . . . . 0

Bits per pixel

| Digit | Frequency | Bits            | TOTAL BITS |
|-------|-----------|-----------------|------------|
| 26    | 1         | 6               | 6          |
| 3     | 2         | 4               | 8          |
| 6     | 1         | 6               | 6          |
| 2     | 4         | 4               | 16         |
| 4     | 2         | 4               | 8          |
| 5     | 1         | 5               | 5          |
| 1     | 8         | 2               | 16         |
| 0     | 45        | 1 <del>45</del> | <u>45</u>  |
|       |           |                 | 110        |

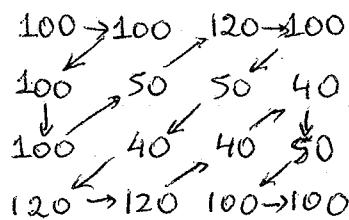
$$bpp = \frac{110}{64} = 1.719 \quad (\text{Much better than } 8 \text{ bpp})$$

(Van Fleet, pg 94)

(gray)

Ex. 5.4 Consider the  $4 \times 4$  image whose intensity matrix

(v)



- Generate the Huffman code for tree for this image
- Write the bit stream for the image using the Huffman code (Zig-zag)
- Compute the bpp of this bit stream

---

 X 

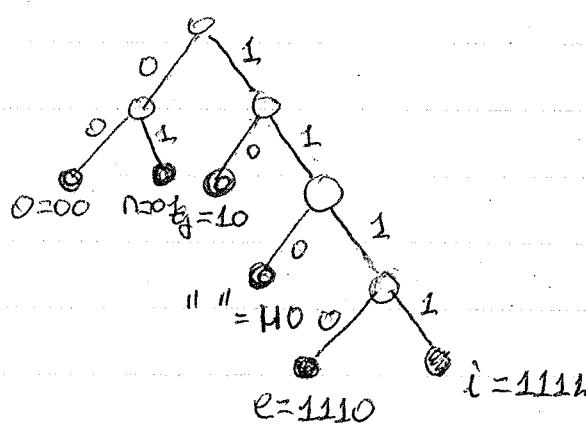
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Ex 5.5 (Van Fleet, pg 95)

Given the Huffman code  $g=10$ ,  $n=01$ ,  $o=00$ ,  
 Space key = 110,  $\ell=1110$ ,  $i=1111$ , draw the Huffman code tree and decode:

100011101101101000111101101101000011110

S/



10 00 1111 01 10 110 10 00 1111 01 10 110 10 00 01 110  
 g 0 i n g " g 0 i " g 0 n e

going, going, gone

Ex 5.6. Given the Huffman code  $D=010$ ,  
 $E=10$ ,  $H=110$ ,  $N=011$ ,  $T=111$ ,  $\underline{\quad}$ =space  
key = 00, draw the Huffman code tree and  
decode the bit stream sequence

111110100010011010

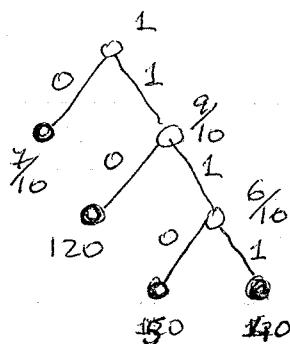
\_\_\_\_\_ x \_\_\_\_\_

THE END



Ex 5.4.

$$\begin{array}{r} 100 - 7 - \frac{7}{16} \\ 120 - 3 - \frac{3}{16} \\ 50 - 3 - \frac{3}{16} \\ 40 - 3 - \frac{3}{16} \\ \hline 16 \end{array}$$



$$\begin{array}{r} 100 - 0 \\ 120 - 10 \\ 50 - 110 \\ 40 - 111 \end{array}$$

000011010011011110101111  
11000

$$\frac{31}{16} \approx 2 \text{ bpp}$$