

SAMPLING OF SIGNALS AND IMAGES.

1. Show that if $f(x) = \frac{1}{T}\chi_{[-\frac{T}{2}, \frac{T}{2}]}(x)$, $x \in \mathbb{R}$, then

$$\mathcal{F}(f)(\xi) = \frac{\sin T\pi\xi}{T\pi\xi}, \quad \xi \in \mathbb{R}.$$

(The function $h(t) = \frac{\sin \pi\xi}{\pi\xi}$ is called the *sinc* (*sinus cardinalis*) function and plays an important rôle in signal processing.)

2. Let $f(x) = e^{-4\pi^2x^2}$, $x \in \mathbb{R}$. Show that

$$\mathcal{F}(f)(\xi) = \frac{1}{2\sqrt{\pi}}e^{-\xi^2/4}, \quad \xi \in \mathbb{R}.$$

3. Show that if $\varphi \in \mathcal{M}$, the mapping U given by $U(\varphi)(x, \xi) = e^{-2\pi i x \xi} \varphi(x, \xi)$, belongs to $\widetilde{\mathcal{M}}$. Moreover, show that $U^*(\widetilde{\varphi})(x, \xi) = e^{2\pi i x \xi} \widetilde{\varphi}(x, \xi)$, where U^* denotes the adjoint to U . (The spaces \mathcal{M} and $\widetilde{\mathcal{M}}$ have been defined in class.)

4. Let V_T be the space of functions in $L^1(\mathbb{R})$ such that $\text{supp } \mathcal{F}(f) \subset [-\frac{T}{2}, \frac{T}{2}]$. Show that if $h_T(x) = \frac{\sin(\pi T x)}{\pi T x}$, then $\left\{h_T(x - \frac{k}{T})\right\}_{k=-\infty}^{k=\infty}$ is an orthogonal basis of V_T . If $f \in V_T \cap L^1(\mathbb{R})$ prove that

$$f\left(\frac{k}{T}\right) = T \int_{-\infty}^{\infty} f(x) h_T\left(x - \frac{k}{T}\right) dx.$$

5. Let $f \in L^1(\mathbb{R})$ and $\text{supp } \mathcal{F}(f) \subset [-\frac{T}{2}, \frac{T}{2}]$. Consider the function

$$F_p(\xi) := \sum_{k=-\infty}^{\infty} \mathcal{F}(f)(\xi + Tk),$$

which is periodic of period T . Show that, as a periodic function, the Fourier series of F_p is

$$\sum_{n=-\infty}^{\infty} \frac{1}{T} f\left(\frac{n}{T}\right) e^{-2\pi i \frac{n}{T} \xi}.$$

6. Given two periodic discrete signals, $f = \{f(n)\}_{n=0}^{N-1}$ and $h = \{h(n)\}_{n=0}^{N-1}$, of period N , the circular convolution is defined as

$$f \circledast h = \sum_{p=0}^{N-1} f(p) h(n-p) \quad n \in \mathbb{Z}.$$

Prove that $f \circledast h = h \circledast f$.

7. Suppose that $\text{supp } \mathcal{F}(f) \subset [-\frac{(n+1)T}{2}, -\frac{nT}{2}] \cup [\frac{nT}{2}, \frac{(n+1)T}{2}]$. Use a similar argument to the one given in the proof for the Shannon Sampling Theorem to show that

$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) \frac{\sin((n+1)\pi(Tx-k)) - \sin(n\pi(Tx-k))}{\pi(Tx-k)}.$$

(Observe that one can recover Shannon Sampling Theorem setting $n = 0$ in the above formula)

8. Denote by $\widehat{f}(k)$ the DFT of a discrete signal of size N (N even). Define $\widetilde{f}\left(\frac{N}{2}\right) = \widetilde{f}\left(\frac{3N}{2}\right) = \widehat{f}\left(\frac{N}{2}\right)$ and

$$\widetilde{f}(k) = \begin{cases} 2\widehat{f}(k) & \text{if } 0 \leq k < N/2 \\ 0 & \text{if } N/2 < k < 3N/2 \\ 2\widehat{f}(k-N) & \text{if } 3N/2 < k < 2N \end{cases}$$

Prove that the discrete signal \widetilde{f} of size $2N$ satisfies $\widetilde{f}(2n) = f(n)$.

9. Let f be the discrete signal of size 4 given by $f = (1, 2, 3, -1)$. Compute the DFT of f using FFT. Check that your result is correct by computing DFT directly.

10. Show that the bidimensional discrete exponentials

$$e_{k,l}(n, m) := e^{\frac{2\pi i k n}{N}} e^{\frac{2\pi i \ell m}{N}}, \quad 0 \leq k, \ell < N,$$

satisfy

$$L_g e_{k,l}(n, m) = e_{k,l}(n, m) \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} g(p, q) e_{k,l}(-p, -q)$$

where $L_g f(n, m) = f \circledast g(n, m)$, for g and f N -periodic bidimensional discrete signals.