Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2015-16 Homework 5
Due: Friday, April 29, 2016

Frame theory.

1. Show that the vectors

$$f_1 = (1, 0), \qquad f_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), \qquad f_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$$

form a tight frame in $\mathbb{R}^2$ with frame constant $3/2$.

2. Let $\{\varphi_k : k = 1, 2, \dots\}$ be a Parseval frame in a Hilbert space $\mathbb{H}$. Show that the following conditions are equivalent:
a) $\{\varphi_k : k = 1, 2, \dots\}$ is an orthonormal basis of $\mathbb{H}$.
b) $\|\varphi_k\| = 1$ for all $k = 1, 2, \dots$.

3. Let $\{\varphi_k : k = 1, 2, \dots\}$ be a frame in a Hilbert space $\mathbb{H}$ with frame bounds A and B. Show that:
a) If for some $k_0 \in \mathbb{N}$ de equality $\|\varphi_{k_0}\|^2 = B$ holds, then $\varphi_{k_0} \in \overline{\text{span}, \{\varphi_k : k \neq k_0\})}^{\perp}$.
b) If for some $k_0 \in \mathbb{N}$, $\|\varphi_{k_0}\|^2 < A$, then $\varphi_{k_0} \in \overline{\text{span}, \{\varphi_k : k \neq k_0\}}$.

4. Let

$$\varphi_k = \begin{pmatrix} a_k \cos\theta_k \\ a_k \sin\theta_k \end{pmatrix}, \qquad k = 1, 2, \dots, M \ (M \geq 2)$$

be vectors in $\mathbb{R}^2$ written in polar coordinates. Prove that $\Phi = \{\varphi_k : k = 1, 2, \dots, M\}$ is a tight frame for $\mathbb{R}^2$ if and only if

$$\sum_{k=1}^{M} a_k^2 \cos 2\theta_k = 0, \quad \text{and} \quad \sum_{k=1}^{M} a_k^2 \sin 2\theta_k = 0\,.$$

(Hint: Use that Φ is a tight frame with constant A if and only if $F = AI$, where F is the frame operator.)

5. Let $\Phi = \{\varphi_k : k = 1, 2, \dots, M\}$ be a frame in $\mathbb{C}^d$, and let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d > 0$ be the eigenvalues of its frame operator F. Prove that

$$\sum_{k=1}^{d} \lambda_k = \sum_{k=1}^{M} \|\varphi_k\|^2.$$

6. Let $\Phi = \{\varphi_k : k = 1, 2, \dots,\}$ be a frame in a separable Hilbert space $\mathbb{H}$, with frame operator F. Since F is a positive, selfadjoint and invertible operator, so is F^{-1}. Its positive square root, denoted by $F^{-1/2}$, is also positive and selfadjoint, and commutes with F. Show that $\Psi = \{\psi_k = F^{-1/2}\varphi_k : k = 1, 2, \dots,\}$ is a Parseval frame.

7. For a given $g \in L^2(\mathbb{R})$, assume that the inequality $\int_{\mathbb{R}} |f(x)g(x)|^2 dx \leq B\|f\|^2$ holds for all $f \in L^2(\mathbb{R})$. Show that $|g(x)|^2 \leq B$ a. e. $x \in \mathbb{R}$.

8. For $g \in L^2(\mathbb{R})$, let $\mathcal{G}(g) = \{M_m T_k g : m, k \in \mathbb{Z}\}$ be a frame for $L^2(\mathbb{R})$. Prove that the frame operator F of the frame $\mathcal{G}(g)$ commutes with modulations M_n and translations T_l.

9. Let

$$g(x) = \begin{cases} 2x+1 & -1/2 \leq x < 0 \\ -2x+1 & 0 \leq x < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Show that $G(g) = \{M_m T_n g : m, n \in \mathbb{Z}\}$ is not a frame for $L^2(\mathbb{R})$.

10. Let $\psi \in L^2(\mathbb{R})$. Suppose that $W(\psi) = \{D_{2^j} T_k \psi : j, k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. Show that its frame operator F commutes with dilations $D_{2^l} f(x) = 2^{\ell/2} f(2^\ell x)\,, \ell \in \mathbb{Z}$.

(*) He sustituido el ejercicio 9 por otro en el que $g \in L^2(\mathbb{R})$. Ahora considera la función

$$g(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x) + \sum_{k \in Z, k \neq 0} \frac{1}{|k|+1} \chi_{[-\frac{1}{2},\frac{1}{2}]}(x-k)\,,$$

que está en $L^2(\mathbb{R})$. Estudia si $\mathcal{G}(g) = \{M_m T_k g : m, k \in \mathbb{Z}\}$ es un marco de $L^2(\mathbb{R})$.