Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2015-16 Homework 3
Due: Monday, April 4, 2016

Wavelets and Multiresolution Analysis.

1. Let $\psi \in L^2(\mathbb{R})$ be defined by $\mathcal{F}(\psi) = \chi_I$, where $I = [-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1]$. Prove that the collection $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ is and orthogonal system in $L^2(\mathbb{R})$ by showing:
a) If $j_1 \neq j_2, j_1, j_2 \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{Z}$, $< \psi_{j_1,k_1}, \psi_{j_2,k_2} >= 0$.
b) If $j \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{Z}$, $< \psi_{j,k_1}, \psi_{j,k_2} >= \delta_{k_1,k_2}$.

2. Show that for the Haar-MRA with scaling function $\varphi = \chi_{[0,1)}$, its low pass filter coefficients are given by $h(0) = h(1) = 1/2$ and $h(k) = 0$ if $k \neq 0, 1$, and the low pas filter is $h(\xi) = e^{-\pi i \xi}\cos(\pi\xi)$.

3. Show that for the Shannon-MRA with scaling function φ given by $\varphi(x) = \dfrac{\sin(\pi x)}{\pi x}, x \neq 0$ and $\varphi(0) = 1$ (recall that $\mathcal{F}\varphi = \chi_{[-1/2,1/2)}$) the associate discrete filter is

$$h(k) = \begin{cases} 0 & \text{if } k = 2\ell, \ell \neq 0 \\ 1/2 & \text{if } k = 0 \\ \frac{(-1)^\ell}{\pi(2\ell+1)} & \text{if } k = 2\ell + 1 \end{cases}$$

Show that the low pass filter of this MRA is

$$h(\xi) = \sum_{k=-\infty}^{\infty} \chi_{[-\frac{1}{4},\frac{1}{4})}(\xi + k)\,.$$

4. Let $h(\xi)$ and $g(\xi)$ be two 1-periodic functions in $L^2([0,1))$ satisfying $g(\xi) = e^{-2\pi i\xi}\overline{h(\xi + \frac{1}{2})}$. If $h(\xi) = \sum_{k=-\infty}^{\infty} h(k)e^{-2\pi ik\xi}$ and $g(\xi) = \sum_{k=-\infty}^{\infty} g(k)e^{-2\pi ik\xi}$, show that

$$g(k) = \overline{h(1-k)}(-1)^{1-k}\,, \quad k \in \mathbb{Z}\,.$$

5. a) Consider the Haar-MRA with scaling function $\varphi = \chi_{[0,1)}$. Show that an orthonormal wavelet associated to this MRA is given by $\psi(x) = \chi_{[0,\frac{1}{2}} - \chi_{[\frac{1}{2},1)}$.

b) For the Shannon MRA with scaling function $\mathcal{F}\varphi = \chi_{[-\frac{1}{2},\frac{1}{2})}$, show that one of this associated orthonormal wavelets is given by

$$\mathcal{F}\psi(\xi) = e^{-\pi i\xi}\chi_{[-1,-\frac{1}{2})\cup[\frac{1}{2},1)}\,.$$

6. a) For a positive integer $p, p > 1$, write the definition of MRA(p), that is a Multiresolution Analysis with dilation factor p.

b) Show that for an MRA(p), there exists a 1-periodic function $h(\xi)$ belonging to $L^2([0,1))$ such that

$$\mathcal{F}\varphi(\xi) = h(\frac{\xi}{p})\mathcal{F}(\frac{\xi}{p}),$$

where φ is the scaling function of the MRA(p). The function h is called the low pass filter of the MRA(p).

7. For and MRA(p), $p > 1$, show that

$$\sum_{s=0}^{p-1} |h(\xi + \frac{s}{p})|^2 = 1 \quad \text{a.e } \xi \in \mathbb{R},$$

where h is the low pass filter of the MRA(p) whose existence is proved in part b) of the previous exercise.

8. Show that the detail coefficients $d_{j-1}(p)$ are computed with the formula

$$d_{j-1}(p) = \sqrt{2} \sum_{k \in \mathbb{Z}} \overline{g(k-2p)} c_j(k),$$

where $\{c_j(k) : k \in \mathbb{Z}\}$ are the coefficients of f at level j and $\{g(k) : k \in \mathbb{Z}\}$ are the high pass filter coefficients.

9. A scaling function of the Haar MRA is $\varphi = \chi_{[0,1]}$ and $\mathcal{F}\varphi(\xi) = e^{-\pi i \xi} \dfrac{\sin(\pi\xi)}{\pi\xi}$. The low pass filter of the Haar MRA is $h(\xi) = e^{\pi i \xi} \cos(\pi\xi)$. Therefore, the following formula must hold:

$$\prod_{j=1}^{\infty} e^{-\pi i \xi / 2^j} \cos(\frac{\pi\xi}{2^j}) = e^{-\pi i \xi} \frac{\sin(\pi\xi)}{\pi\xi}.$$

Show this formula directly using the trigonometric relation $\sin(2\alpha) = 2(\sin\alpha)(\cos\alpha)$.

10. Let $h(\xi) = e^{-3\pi i \xi} \cos(3\pi\xi)$. Clarly, h is a continuous function on $\mathbb{R}$, it is one periodic, and $h(0) = 1$.

a) Show that

$$|h(\xi)|^2 + h(\xi + 1/2)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}.$$

b) Define φ by $\mathcal{F}\varphi(\xi) = \prod_{j=1}^{\infty} h(\xi/2^j)$. Show that $\varphi = \dfrac{1}{3}\chi_{[0,3]}$. (Observe that the integer translates of φ do not form an orthogomal system in $L^2(\mathbb{R})$.)