Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2014-15 Homework 5
Due: Thursday, May 14, 2015

Coding and Entropy. Biorthogonal wavelets.

1. Show that for any source of information X with N symbols

$$0 \leq \mathcal{E}(X) \leq \log_2 N,$$

where $\mathcal{E}(X) = \sum_{k=1}^{n} p_k \log_2 \frac{1}{p_k}$ is the Shannon entropy of X for the probabilities p_k, $k = 0, 1, 2, \ldots, N$. (Hint: Use Lagrange multipliers to show that the maxima of $\mathcal{E}(X)$ in the region $0 \leq p_k \leq 1, k = 1, 2, \ldots, N$, with the restriction $\sum_{k=1}^{N} p_k = 1$ is attained at $p_k = 1/N, k = 1, 2, \ldots, N$.

2. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ be a source of information whose probabilities are

$$p_1 = 0.49,\ p_2 = 0.26,\ p_3 = 0.12, p_3 = 0.04,\ p_5 = 0.04,\ p_6 = 0.03, p_7 = 0.02.$$

a) Compute the entropy of X. Build a binary Huffman code $\mathcal{C}$ for these probabilities and compute $\mathcal{A}_{\mathcal{C}}(X)$.

b) Suppose that the symbols of X are codify with a ternary code that takes values 0, 1, and 2. The code can now be represented in a ternary tree. Extend Huffman algorithm to find a ternary tree for X having the prefix condition and compute the average length of its codified words.

3. The quantize coefficients of an 8×8 block of an image are

$$\begin{bmatrix} -26 & -3 & -6 & 2 & 2 & -1 & 0 & 0 \\ 0 & -2 & -4 & 1 & 1 & 0 & 0 & 0 \\ -3 & 1 & 5 & -1 & -1 & 0 & 0 & 0 \\ -4 & 1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a binary Huffman code to represent these symbols.

4. For a discrete signal $x = \{x(n) : n \in \mathbb{Z}\}$ use the notation $x(\xi) = \sum_{n \in \mathbb{Z}} x(n) e^{-2\pi i n \xi}$ to denote its Fourier series.

a) Let $\underline{x} = \{x(2n) : n \in \mathbb{Z}\}$ be the down sampling by 2 of the signal $x = \{x(n) : n \in \mathbb{Z}\}$. Show that

$$\underline{x}(2\xi) = \frac{1}{2}[x(\xi) + x(\xi + \frac{1}{2})].$$

b) Let $\check{x}$ be the upsampling by 2 of the signal $x = \{x(n) : n \in \mathbb{Z}\}$, that is, $\check{x}(n) = x(p)$ if $n = 2p$ and $\check{x}(n) = x(0)$ if $n = 2p + 1$. Show that $\check{x}(\xi) = x(2\xi)$.

5. Choose real finite filters h and $\widetilde{h}$ such that

$$\overline{h(\xi)}\,\widetilde{h}(\xi) + \overline{h(\xi+1/2)}\,\widetilde{h}(\xi+1/2)\,, \qquad a.e.\ \xi \in \mathbb{R}.$$

Show that if a is a non-zero real number and ℓ is an integer,

$$g(\xi) = ae^{-2\pi i(2\ell+1)\xi}\,\overline{\widetilde{h}(\xi+\frac{1}{2})}\,, \quad \text{and} \quad \widetilde{g}(\xi) = \frac{1}{a}e^{-2\pi i(2\ell+1)\xi}\,\overline{h(\xi+\frac{1}{2})}\,,$$

then $(h, \widetilde{h}, g, \widetilde{g})$ is a filter bank for perfect reconstruction.

6. a) Let $h(\xi) = \sum_{n\in\mathbb{Z}} h(n)e^{-2\pi in\xi}$ and $\tilde{h}(\xi) = \sum_{n\in\mathbb{Z}} \tilde{h}(n)e^{-2\pi in\xi}$ be two filters. Show that the following conditions are equivalent:

$i)$ $\tilde{h}(\xi)\,\overline{h(\xi)} + \tilde{h}(\xi+\frac{1}{2})\,\overline{h(\xi+\frac{1}{2})} = 1 \quad a.e.\ \xi \in \mathbb{R}.$ $\qquad ii)$ $\sum_{k\in\mathbb{Z}} \tilde{h}(k)\,\overline{h(k-2\ell)} = \frac{1}{2}\delta_{0,l}\,, \qquad \ell \in \mathbb{Z}.$

b) Consider $\tilde{h}(\xi) = \frac{1}{4}(e^{-2\pi i\xi} + 2 + e^{2\pi i\xi})$ a real, finite, symmetric filter. Find $h(\xi) = \sum_{k=-2}^{k=2} h(k)e^{2\pi ik\xi}$, real, symmetric filter of length 5 such that $h(1/2) = 0$ and satisfies i) of part a). (This is the CDF(5,3) filter.)

7. Let $\varphi, \tilde{\varphi} \in L^2(\mathbb{R})$. Prove that the following conditions are equivalent:

$$i)\ \langle \varphi(\cdot - k), \tilde{\varphi}(\cdot - \ell)\rangle = \delta_{k,l} \quad \text{for all } k, \ell \in \mathbb{Z}\,,$$

$$ii)\ \sum_{k\in\mathbb{Z}} \mathcal{F}(\varphi)(\xi+k)\overline{\mathcal{F}(\tilde{\varphi})(\xi+k)} = 1, \quad \text{a.e. } \xi \in \mathbb{R}.$$

8. Show that the low pass filter associated to the spline $\tilde{\varphi}^{(n)} = \Delta^n(x + N_n)$ is given by

$$\tilde{h}^{(n)}(\xi) = e^{-i\epsilon_n\pi\xi}(\cos\pi\xi)^{n+1}.$$

9. Take $n = 2\tilde{\ell}$ even natural number and consider the low pass filter

$$\tilde{h}^{(\tilde{\ell})}(\xi) = e^{-i\pi\xi}(\cos\pi\xi)^{2\tilde{\ell}+1}.$$

Show that for any $\ell = 1, 2, 3, \ldots$, a solution of the equation

$$i)\ h(\xi)\,\overline{\tilde{h}^{(\tilde{\ell})}(\xi)} + h(\xi+\frac{1}{2})\,\overline{\tilde{h}^{(\tilde{\ell})}(\xi+\frac{1}{2})} = 1 \qquad \xi \in \mathbb{R}.$$

is

$$h(\xi) := h^{(\ell)}(\xi) = e^{-i\pi\xi}(\cos\pi\xi)^{2\ell+1}\sum_{n=0}^{\ell+\tilde{\ell}} \binom{\ell+\tilde{\ell}+n}{n}(\sin\pi\xi)^{2n}\,.$$

(Hint: Follow the steps that lead to the solution of equation i) in the case n odd.)

10. Find the coefficients of the filter CDF(9,7) used by JPEG2000. (Take $\tilde{\ell} = 3$ and $\ell = 1$.) Use symbolic computation to do the calculations. Draw the graphs of $h(\xi)$ and $\tilde{h}(\xi)$ in $[-1, 1]$.