Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2014-15 Homework 4
Due: Friday, April 25, 2015

## Explicit constructions of wavelets.

**1.** Show that

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi+k)^2} = \frac{\pi^2}{(\sin \pi\xi)^2},$$

using that $\{\chi_{[0,1]}(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$.

**2.** It has been proved in class that

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi+k)^{(N+1)}} = \left(\frac{\pi}{\sin \pi\xi}\right)^{(N+1)} P_N(\pi\xi).$$

Show that

$$P_2(\xi) = \cos\xi$$
$$P_3(\xi) = \frac{2}{3} + \frac{1}{3}\cos(2\xi)$$
$$P_4(\xi) = \frac{1}{3}\cos^2\xi + \frac{2}{3}\cos^3\xi$$
$$P_5(\xi) = \frac{1}{30}\cos^2(2\xi) + \frac{13}{30}\cos(2\xi) + \frac{8}{15}.$$

**3.** Show the following relation for the basic splines:

$$\Delta^n(x) = \frac{1}{2^n}\sum_{k=0}^{n+1} \binom{n+1}{k} \Delta^n(2x-k).$$

(Hint: Compute $\frac{\mathcal{F}(\Delta^n)(\xi)}{\mathcal{F}(\Delta^n)(\xi/2)}$.)

**4.** a) Show that if $n$ is odd, the spline scaling function $\varphi^n$ is even.
b) Show that if $n$ is even, the spline scaling function $\varphi^n$ is even with respect to $x = 1/2$.

**5.** A scaling function of the Haar MRA is $\varphi = \chi_{[0,1]}$ and $\mathcal{F}\varphi(\xi) = e^{-\pi i\xi}\frac{\sin(\pi\xi)}{\pi\xi}$. The low pass filter of the Haar MRA is $h(\xi) = e^{\pi i\xi}\cos(\pi\xi)$. Therefore, the following formula must hold:

$$\prod_{j=1}^{\infty} e^{-\pi i\xi/2^j}\cos(\frac{\pi\xi}{2^j}) = e^{-\pi i\xi}\frac{\sin(\pi\xi)}{\pi\xi}.$$

Show this formula directly using the trigonometric relation $\sin(2\alpha) = 2(\sin\alpha)(\cos\alpha)$.

**6.** Let $h(\xi) = e^{-3\pi i\xi}\cos(3\pi\xi)$. Clarly, $h$ is a continuous function on $\mathbb{R}$, it is one periodic, and $h(0) = 1$.

a) Show that

$$|h(\xi)|^2 + h(\xi + 1/2)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}.$$

b) Define $\varphi$ by $\mathcal{F}\varphi(\xi) = \prod_{j=1}^{\infty} h(\xi/2^j)$. Show that $\varphi = \frac{1}{3}\chi_{[0,3]}$. (Observe that the integer translates of $\varphi$ do not form an orthogomal system in $L^2(\mathbb{R})$.)

**7.** Let $c_k$ be given by

$$\frac{1}{c_k} = \int_0^{1/2} (\sin 2\pi x)^{2k+1} dx > 0,$$

and

$$g_k(\xi) = 1 - c_k \int_0^{\xi} (\sin 2\pi x)^{2k+1} dx\,.$$

a) Show that

$$g_0(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^2.$$

b) Show that

$$g_1(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^4 (2 - \cos 2\pi\xi)\,.$$

(Hint: Write $\sin^2 x = (1 + \cos x)(1 - \cos x)$ and integrate by parts.)

**8.** Find the low pass filter coefficients of Daubechies ${}_2\psi$ orthonormal wavelet using the polynomial

$$g_1(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^4 (2 - \cos 2\pi\xi)\,,$$

found in the previous exercise.

**9.** (**2 puntos**) Let $c_k$ be given by

$$\frac{1}{c_k} = \int_0^{1/2} (\sin 2\pi x)^{2k+1} dx > 0,$$

and

$$g_k(\xi) = 1 - c_k \int_0^{\xi} (\sin 2\pi x)^{2k+1} dx\,.$$

Show, integrating by parts, that

$$g_k(\xi) = \left|\frac{1 + e^{-2\pi i\xi}}{2}\right|^{2k+2} P_k(\xi)\,,$$

where

$$P_k(\xi) = \frac{1}{2^k} \sum_{\ell=0}^{k} \binom{2k+1}{k-\ell} (1 + \cos 2\pi\xi)^{\ell} (1 - \cos 2\pi\xi)^{k-\ell}$$

is and even trigonometric polynomial with real coefficients.