Universidad Autónoma de Madrid
Master in Mathematics and Applications
Wavelets and Signal Processing - 2014-15 Homework 3
Due: Thursday, March 26, 2015

Wavelets and Multiresolution Analysis.

1. Let $\psi \in L^2(\mathbb{R})$ be defined by $\mathcal{F}(\psi) = \chi_I$, where $I = [-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1]$. Prove that the collection $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ is and orthogonal system in $L^2(\mathbb{R})$ by showing:
a) If $j_1 \neq j_2, j_1, j_2 \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{Z}$, $< \psi_{j_1,k_1}, \psi_{j_2,k_2} >= 0$.
b) If $j \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{Z}$, $< \psi_{j,k_1}, \psi_{j,k_2} >= \delta_{k_1,k_2}$.

2. Let $E \subset \mathbb{R}$ be a measurable set such that
a) $\{E + k : k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$,
b) $\{2^j E : j \in \mathbb{Z}\}$ is a partition of $\mathbb{R} \setminus \{0\}$.

i) Prove that $\{e^{2\pi i k\xi} : k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(E)$. (Hint: Since $\{E+k : k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$, there exist $k_l \in \mathbb{Z}, l \in Z$, and a partition $\{E_\ell : \ell \in \mathbb{Z}\}$ of E such that $\{E_\ell + k_\ell : \ell \in \mathbb{Z}\}$ is a partition of $[0, 1)$.)

ii) Let $\psi \in L^2(\mathbb{R})$ be defined by $\mathcal{F}\psi = \chi_E$. Show that ψ is an orthonormal wavelet for $L^2(\mathbb{R})$. (Hint: to prove completeness imitate the proof of the completeness of the Shannon wavelet.)

3. i) (Journé) Let

$$E_J = [-\frac{16}{7}, -2) \cup [-\frac{1}{2}, -\frac{2}{7}) \cup [\frac{2}{7}, \frac{1}{2}) \cup [2, \frac{16}{7})\,.$$

Show that ψ_J given by $\mathcal{F}\psi_J = \chi_{E_J}$ is an orthonormal wavelet for $L^2(\mathbb{R})$. (Hint: Use the previous exercise.)

ii) (Lemarié) Let

$$E_L = [-\frac{4}{7}, -\frac{2}{7}) \cup [\frac{2}{7}, \frac{3}{7}) \cup [\frac{12}{7}, \frac{16}{7})\,.$$

Show that ψ_L given by $\mathcal{F}\psi_J = \chi_{E_L}$ is an orthonormal wavelet for $L^2(\mathbb{R})$. (Hint: Use the previous exercise.)

4. Show that for the Haar-MRA with scaling function $\varphi = \chi_{[0,1)}$, its low pass filter coefficients are given by $h(0) = h(1) = 1/2$ and $h(k) = 0$ if $k \neq 0, 1$, and the low pas filter is $h(\xi) = e^{-\pi i\xi}\cos(\pi\xi)$.

5. Show that for the Shannon-MRA with scaling function φ given by $\varphi(x) = \dfrac{\sin(\pi x)}{\pi x}\,, x \neq 0$ and $\varphi(0) = 1$ (recall that $\mathcal{F}\varphi = \chi_{[-1/2,1/2)}$) the associate discrete filter is

$$h(k) = \begin{cases} 0 & \text{if } k = 2\ell, \ell \neq 0 \\ 1/2 & \text{if } k = 0 \\ \frac{(-1)^\ell}{\pi(2\ell+1)} & \text{if } k = 2\ell + 1 \end{cases}$$

Show that the low pass filter of this MRA is

$$h(\xi) = \sum_{k=-\infty}^{\infty} \chi_{[-\frac{1}{4},\frac{1}{4})}(\xi + k) .$$

6. Let $h(\xi)$ and $g(\xi)$ be two 1-periodic functions in $L^2([0,1))$ satisfying $g(\xi) = e^{-2\pi i\xi}\overline{h(\xi + \frac{1}{2})}$. If $h(\xi) = \sum_{k=-\infty}^{\infty} h(k)e^{-2\pi ik\xi}$ and $g(\xi) = \sum_{k=-\infty}^{\infty} g(k)e^{-2\pi ik\xi}$, show that

$$g(k) = \overline{h(1-k)}(-1)^{1-k}, \quad k \in \mathbb{Z} .$$

7. a) Consider the Haar-MRA with scaling function $\varphi = \chi_{[0,1)}$. Show that an orthonormal wavelet associated to this MRA is given by $\psi(x) = \chi_{[0,\frac{1}{2}} - \chi_{[\frac{1}{2},1)}$.

b) For the Shannon MRA with scaling function $\mathcal{F}\varphi = \chi_{[-\frac{1}{2},\frac{1}{2})}$, show that one of this associated orthonormal wavelets is given by

$$\mathcal{F}\psi(\xi) = e^{-\pi i\xi}\chi_{[-1,-\frac{1}{2})\cup[\frac{1}{2},1)} .$$

8. a) For a positive integer $p, p > 1$, write the definition of MRA(p), that is a Multiresolution Analysis with dilation factor p.

b) Show that for an MRA(p), there exists a 1-periodic function $h(\xi)$ belonging to $L^2([0,1))$ such that

$$\mathcal{F}\varphi(\xi) = h(\frac{\xi}{p})\mathcal{F}(\frac{\xi}{p}) ,$$

where φ is the scaling function of the MRA(p). The function h is called the low pass filter of the MRA(p).

9. For and MRA(p), $p > 1$, show that

$$\sum_{s=0}^{p-1} |h(\xi + \frac{s}{p})|^2 = 1 \quad \text{a.e } \xi \in \mathbb{R} ,$$

where h is the low pass filter of the MRA(p) whose existence is proved in part b) of the previous exercise.

10. Show that the detail coefficients $d_{j-1}(p)$ are computed with the formula

$$d_{j-1}(p) = \sqrt{2}\sum_{k\in\mathbb{Z}} \overline{g(k-2p)}c_j(k) ,$$

where $\{c_j(k) : k \in \mathbb{Z}\}$ are the coefficients of f at level j and $\{g(k) : k \in \mathbb{Z}\}$ are the high pass filter coefficients.