

## Tessellations

Students build tessellations of one or more regular, irregular, or convex polygons, and they look at nonperiodic tilings. After students have learned about the angle relationships in tilings, they are asked to describe analogies they see between tessellations and polyhedra.

### Goals

- To understand and create tessellations and tilings
- To learn concepts and notations important for understanding and discussing polyhedra

### Prerequisites

Students should be familiar with the sums of the angles in triangles and other polygons and the angles of regular polygons (see Unit 1).

### Notes

For a fuller introduction to tessellations, see Unit 7 in *Geometry Labs*, by Henri Picciotto, and the software *Kaleidomania*, by Kevin Lee (both available from Key Curriculum Press). The ultimate reference work on this subject is *Tilings and Patterns*, by Branko Grünbaum and G. C. Shephard.

### 8.1 Basic Tessellations

Students can get a substantial head start on this unit by spending some time on the Challenge. Acknowledge any discoveries, and have students share them with their classmates. Once no more progress is being made, hand out the activity page and have students start on the more guided part of the activity.

A key concept here is that, in a tessellation, the angles around a vertex must add up to 360 degrees. This helps answer most of the questions in this activity. (See the Answers for how.) However, while this condition is necessary, it is not sufficient. For example, the arrangement 10-5-5 works out around a single point but cannot be extended to tile the plane.

For one of the Archimedean tessellations, a Zome model requires green struts.

## 8.2 Nonperiodic Tilings

This activity is optional; it does not relate to the rest of this book in an essential way. Nevertheless, it is an interesting example of contemporary mathematics that the Zome System makes accessible. More information can be found in Martin Gardner's book *Penrose Tiles to Trapdoor Ciphers*.



Teacher Notes

# 8.1

## Basic Tessellations

### Challenge

Create some tessellations. Record them in your notebook. Can you find some with

- only one type of polygon?
  - more than one type of polygon?
  - regular polygons?
  - nonregular polygons?
  - nonconvex polygons?
- 

A *tessellation* is a pattern of geometric figures that covers the plane and repeats infinitely in two dimensions, with no gaps and no overlaps.

Two well-known tessellations are the square pattern of the checkerboard and the hexagonal pattern of the honeycomb or chicken wire. Both patterns are widely used for tiling floors.

When you are building tessellations, two or three repetitions in each direction are enough to get the idea in each case. Do not spend too much time trying to extend any pattern off to infinity! You can create tessellations using the Zome System, or by using graph paper or a template with geometric figures if you have one. Or you can fold a piece of paper in half three times and cut out eight copies of the polygon you want to experiment with. The advantage of using the Zome System is that you can get equal angles and sides with great accuracy, and you can hold up your tessellations to show your classmates. The disadvantage is that not all angles and side lengths are available.

1. Find a scalene triangle tessellation.

**Q1** Sketch your tessellation, indicating the angles around a vertex.

2. Find a tessellation using an arbitrary quadrilateral.

**Q2** Sketch your tessellation, indicating the angles around a vertex.

3. Find a pentagon that will tessellate the plane.

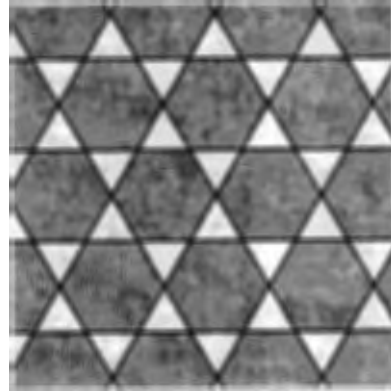
**Q3** Sketch your tessellation, indicating the angles around a vertex.

The simplest tessellations are composed of a single type of regular polygon. These are called the *regular tessellations*.

4. Find all the regular tessellations.

**Q4** Sketch your tessellations, indicating the angles around a vertex.

There are eight tessellations that involve two or more types of regular polygons, with the same arrangement at each vertex. These are called *Archimedean tessellations*, by analogy with the *Archimedean polyhedra*, which will be discussed in Unit 12. The one shown here is  $(3, 6, 3, 6)$ , which says that each vertex is surrounded by alternating equilateral triangles and regular hexagons. It can be constructed with the Zome System, and so can  $(3, 3, 3, 3, 6)$ . If you have green struts, then  $(4, 8, 8)$  can also be constructed, but to find others you will have to use some other tool. The zomeball does not allow them, since squares are in different planes than triangles and hexagons.



An Archimedean tessellation

5. Find some examples of Archimedean tessellations.

Q5 What analogies do you see between tessellations and polyhedra? What differences do you see?

# 8.2

## Nonperiodic Tilings

### Challenge

Fill the plane with polygons in a way that no combination repeats.

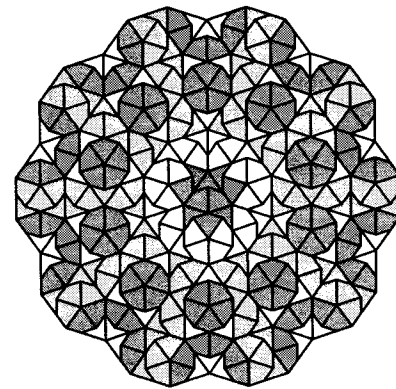
1. Find a parallelogram (or a rhombus) that can be used all by itself in more than one type of tessellation.

**Q1** Notice that a regular hexagon can be dissected into three 60-degree rhombi. Does this suggest any new rhombic tessellations?

Tessellations work by exact repetition. But the plane can be filled in such a way that the pattern does not exactly repeat. This means that two different small regions might be the same, but if large enough neighborhoods are included around them, the large neighborhoods are always different. This is called a *nonperiodic tiling*.

One way to achieve a nonperiodic tiling is to use the tessellation of a regular hexagon as a starting point. Then you dissect a hexagon into three rhombi in two different ways. (Turn one 60 degrees to get the other.) If you start with the tessellation of regular hexagons and dissect them all into rhombi, each time choosing at random one of the two choices, large regions will never be exactly the same. There will be vertices with 3, 4, 5, and 6 edges.

In the 1960s, the British mathematician Roger Penrose (b. 1931) discovered ways to cover the plane with polygons in which the pattern is sure never to repeat, without needing to introduce randomness. A nonperiodic tiling of “kites and darts” is shown here. The Zome System allows these patterns.



Nonperiodic Penrose tiling

**Q2** From the figure, determine the interior angles of Penrose’s kites and darts.

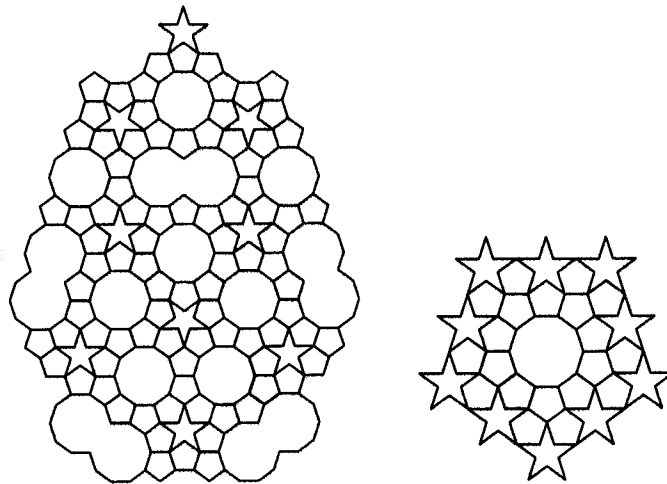
2. Create a tessellation using Penrose’s kites and darts.
3. Create a nonperiodic tiling using Penrose’s kites and darts.

Penrose also discovered that two rhombi, with 36- and 72-degree angles, can be used to cover the plane in ways that do not repeat.

4. Create a tessellation using Penrose’s rhombi.
5. Create a nonperiodic tiling using Penrose’s rhombi.

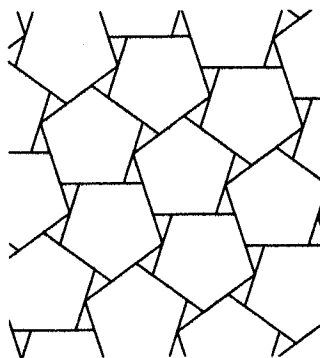
## Explorations 8

- A. **Nonconvex Tiles** Make a Zome tessellation with a nonconvex quadrilateral.
- B. **Pentagon Tiles** Make a Zome tessellation that uses regular pentagons and just one other type of polygon. (The other type need not be regular.)
- C. **Kepler's Tessellations** The German astronomer Johannes Kepler (1571–1630), who discovered that planets have elliptical orbits, was also interested in the problem of tessellations that involve pentagons. The figures replicate some patterns he published in the early 1600s, involving regular pentagons, regular decagons, and other polygons. Make one of these with the Zome System.



Copies of Kepler's tessellations

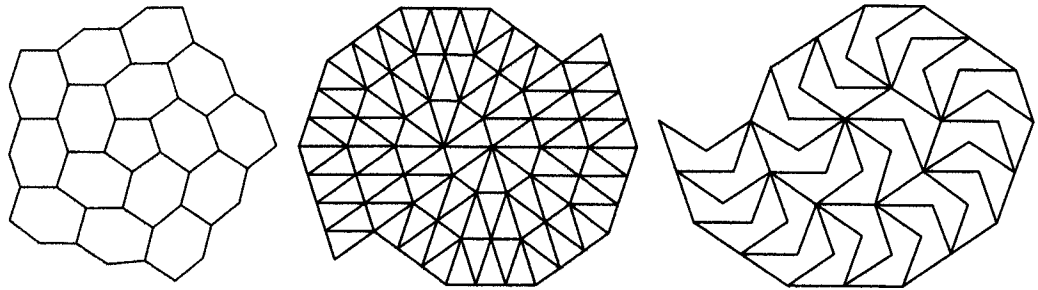
- D. **Pentagons and Triangles** The figure below shows a pattern with regular pentagons and isosceles triangles. Each pentagon touches six surrounding pentagons. Make this pattern with the Zome System.



Pentagon and triangle tessellation

**Explorations 8 (continued)**

- E. Escher** M. C. Escher modified the edges of tessellations to make repeating human and animal forms that cover the plane. Explore by making your own Escher-like drawings, starting with any tessellation.
- F. Spiral Tessellations** The figures below show some interesting spiral tessellations. In the first, there is one regular pentagon surrounded by identical irregular equilateral hexagons. Explain how to extend it to infinity. The second is a double spiral composed of isosceles triangles. If you slide the bottom half to the left by the length of the side of the triangle, you would have a pattern with ten-fold symmetry. The third is a one-arm spiral, using a concave equilateral pentagon. Construct each of these with the Zome System.



Three spiral tessellations