Superficies MANUAL

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1. Introduction

Superficies (that in Spanish means "Surfaces") is a program for visualizing the intrinsic and extrinsic geometry of surfaces in $\mathbb{R}^3$ and also the geometry of a rectangular domain in $\mathbb{R}^2$ endowed with a pseudo-Riemannian metric. It aims to be a teaching tool, but you are free to use it as you like, from designing a pot to analyzing the form of a singularity.

For drawing surfaces, you write down the equations that define the surface and the domain or range where you need to compute it; then, **Superficies** draws the surface in a realistic manner. After drawing it, you can interact with **Superficies** for obtaining geometrical data. That is, as you move the mouse, **Superficies** shows in real time the coordinates and curvatures of the surface point at which the mouse is walking. Changing the menu option, you can obtain asymptotic lines and directions, curvature lines and directions, geodesics, geodesic balls, exponential mapping, lines of constant geodesic curvature, and (with some luck) a geodesic between two points fixed by you (not always minimizing the distance, though). For pseudo-Riemannian planes and for surfaces in parametric form, you may also draw the parallel transport of a vector along the surface. You can get a wireframe view of the surface, and move it in real time; this allows for positioning it in the best way for a new rendering.

You may change the ambient geometry by choosing "Minkowski space", so that from then on, **Superficies** assumes that the ambient $\mathbb{R}^3$ has the metric:

$$dx^2 + dy^2 - dz^2.$$

Accordingly, you may also get then the "light rays" on the surface, that is the lines on the surface that have null tangents.

You can define the surface in three manners, implicit and parametric and tube. In the implicit way, you must provide **Superficies** with a function in $\mathbb{R}^3$, say $f(x, y, z)$, and fix the range as a cube in $\mathbb{R}^3$ with edges parallel to the axes where the surface $f = 0$ is to be drawn.

In parametric form, you write down three functions $x(u, v), y(u, v), z(u, v)$ that parametrize the surface, and fix the domain in the u, v plane, which must be a rectangle with edges parallel to the axes.

If you choose "Tube..." in the menu, then you must write down the equations $x(t), y(t), z(t)$ of a curve, and then another function $r(t)$. The program will draw a surface generated by circles $\mathbf{c}(t)$. The circle $\mathbf{c}(t)$ is centered at the point $\mathbf{p}(t) = (x(t), y(t), z(t))$, has radius $r(t)$ and lies in the plane orthogonal to the tangent to the curve at that point $\mathbf{p}(t)$.

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When you want to see the geometry of a pseudo-Riemannian rectangle U in $\mathbb{R}^2$, you give the functions $E(x,y), F(x,y), G(x,y)$, and fix the domain U as a rectangle as before. Then, **Superficies** assumes that you have on U a pseudo-Riemannian metric

$$g = Edx^2 + 2Fdxdy + Gdy^2,$$

and draws the rectangle colored according to the curvature of g. As in the case of surfaces, you may get in real time the coordinates and curvature of the point where the mouse is walking, and the light rays, geodesics, etc. Of course, you do not have curvature or asymptotic lines because they belong in extrinsic geometry, but you have instead a real time parallel transport.

You may save the surface as a bitmap, print it, etc.

2. Defining the surfaces

Double-click or "Open" **Superficies** in the usual way. You will see a window with two drawing panes for a parametrized surface. In the "File" menu you will find the four main options "Parametric Surface...", "Implicit Surface...", "Pseudo-Riemannian Plane..." and "Tube...". When you select one of them, a window opens where you will edit the equations defining the surface. You may have only one window open at a time, so that you will be asked to save the previous one in order to open a window of different type.

Superficies is rather forgiving about the input of equations. That is, it knows about different spellings of some functions, say "arctan" = "atan". It does not need the times "*" symbol between factors nor even a blank, except in very rare cases; however, if you feel more secure, put a blank space between factors. It recognizes the string "pi" as 3.1415... but not the Greek letter π itself. But it does need that the arguments of functions be put between parentheses. Besides the functions:

plus (+), minus (–), times (*), divide (/), and power (a^b or a**b),

you can use the following functions:

sqr (square), sqrt (square root), sin, cos, tan, cot, arcsin (= asin), arccos (= acos), arctan (= atan), exp, ln (=log), sinh, cosh, tanh, arcsinh (= asinh), arccosh (= acosh), arctanh (= atanh), abs (absolute value).

Suppose that you did select the default, "Parametric Surface..." At the left of the window bottom you see three edit spaces labeled "x(u,v) = ", etc. By default, they show formulas that you can change. For instance, write

$$\begin{aligned} x(u,v) &= \cos(\text{u})\sin(\text{piv}), \\ y(u,v) &= \sin(\text{u})\sin(\text{pi v}), \\ z(u,v) &= 2\cos(\text{piv}). \end{aligned} \tag{2.1}$$

Then, fix the domain writing

$$0 < u < 2\text{pi}, \quad -1 < v < 1$$

Click in the button "Draw". After some seconds, **Superficies** will show the surface in the right panel of the window. You will see another blank panel to the left that represents the domain of the coordinates "u", and "v". Both panels are "synchronized", that is, with the exception of moving the surface, which occurs in the right panel and “Parallel Transport” which occurs in the left one, everything that happens in one of them is mapped in real time on the other panel by means of the map $\mathbf{p}(u, v) = (x(u, v), y(u, v), z(u, v))$ or its inverse.

Now, close the window, and select “Implicit Surface...” in the “File” menu, enter a formula for a function on the variables “x”, “y”, “z”, fix the cube in $\mathbb{R}^3$ where the surface is to be drawn and click the “Draw” button. If you are tired of waiting, press the “Stop” button that terminates the task, or click in another application window and leave **Superficies** computing in the background.

Close the window, select “Tube...”, enter formulas for the functions “x(t)”, “y(t)”, “z(t)”, and “r(t)” and click in “Draw”. Once the tube is drawn, it will behave for all purposes as a parametric surface. This shall be understood in the following without further comment.

Now, select “Pseudo-Riemannian Plane...”, edit if you like the functions on the variables “x”, “y” that define the metric, and the rectangle in $\mathbb{R}^2$ that defines the domain, and click “Draw”. The plane will be colored according to the curvature.

Superficies will display in the panel “Information” some messages that can be of use to you, for instance when it does not agree with the input of a function or of the domain. For reasons of space, in the Parametric or Tube cases, the information is shown in the status panel at the bottom of the window.

3. About the rendering

For pseudo-Riemannian planes, **Superficies** draws the rectangle in a color that represents the curvature and whether the metric is definite or not at each point. That is, when the metric is definite (Riemannian) the color is reddish for the points with positive curvature, yellow for negative and white for near zero curvature; when the metric is not definite (pseudo-Riemannian proper) the color is blue for positive, purple for negative and white for near zero curvature. Points where the metric is degenerate are skipped so that they appear white.

For implicit and parametric surfaces, **Superficies** draws each side of the surface in a different color, golden or turquoise by default, but you can pick other colors. It is assumed that the surface is lighted by an ambient light and by a pointlike light source that casts shadows in a realistic manner.

For parametric surfaces, there is sometimes a problem when the program cannot decide what part of the surface is nearest to the viewer. Then, it might show in an apparently random manner adjacent little triangles golden or turquoise. Usually this is due to “repetitions”, that is the surface passes several times by the same points. The only thing that can be done then is to limit the domain for avoiding those repetitions or to increase the grid density in the bottom right corner of the window (the maximum density is 127).

If after a time of drawing lines on the surface you want to wipe out all them, select “Refresh” in the “File” menu. In order to redraw the window or print them with good resolution, these lines are stored in memory and may occupy a large part of it. Thus, I have put a limit to this memory allocation whose trespassing may pop up on some occasion as a warning. When you “Refresh” the surface, the program releases all the allocated space and begins storing lines anew.

When a surface is already drawn you can use the menu “Save Drawing to Clipboard” to send to the clipboard a copy of the surface panel in bitmap format; in the parametric case, if this is done with the shift key pressed, then the parameter panel will be copied instead of the surface panel.

When the viewing mode of a surface is not a wireframe, the "Print" menu leads to a dialog with two parts. On the right, you may select the resolution of the picture (the number of pixels in each direction) that will be sent to the printer. The "As in Screen" option leads to an immediate stream to the printer, whereas the other ones need to be computed from scratch in the background. The time and RAM needed increases roughly with the square of the resolution, so that you have the option to "Stop" the process in the meanwhile.

When the viewing mode of the surface is the standard one, on the left part of the dialog you may choose what to send to the printer: the surface without any lines; the surface with the lines that you check in the bottom part; or in the parametric case, only the lines in the parameter space that are checked. All these lines are always printed with the best resolution.

4. Preferences

In the menu "Preferences" you have first the decision whether consider the ambient $\mathbb{R}^3$ as Euclidean or Minkowskian. In the first case, the inner product is the usual, that is $dx^2 + dy^2 + dz^2$, and in the second it is $dx^2 + dy^2 - dz^2$, so that the coordinate "z" act as "time" in the relativistic sense.

When drawing lines you can choose among three options. The lines can be drawn in both sides of the surface, or you can choose one of them: "Draw on Gradient Side" draws lines in the side towards which the gradient of $f(x, y, z)$ (implicit surfaces) or the normal associated to the parametrization (parametric) is pointing. "Draw on the Other Side" speaks by itself.

In parametric or implicit surfaces, you may choose to see the cube that defines the range (implicit) or contains tightly the surface (parametric). In pseudo-Riemanian planes and parametric surfaces, you have the item "Put Unit Circle", so that after selecting it you may see, as you move the mouse, the unit circle of the tangent space (for pseudo-Riemannian points, the "circle" includes also the vectors with square length equal to -1, so that it consists of two hyperbolae).

When the window is a surface, you have a hierarchical submenu headed "View Mode". On it you have the following options: "Colored Surface, "Colored Stereo Pair", "Movable", "Wireframe Mono", "Wireframe Stereo", "Wireframe Stereo Pair". The first one is the default realistic view. The second draws a pair of colored surfaces that can be looked with crossed eyes to see the surface as three-dimensional.

The last three options are wireframes and all three can be moved with the mouse. The first draws a simple wireframe, the second a pair to be looked with a pair of green-red eyeglasses, and the third to be looked with crossing eyes. By dragging the mouse you can rotate the surface; pressing $<$ Control $>$ and dragging you can translate it; and zoom it by pressing $<$ Shift $>$ and dragging up or downwards.

If after moving a wireframe option you press the button "Redraw" you will get a rendering of the surface in that position in the default realistic mode.

Finally, we have the "Movable" view mode. After selecting it, you can move directly the surface with the mouse in the manner described above for wireframes.

You have two more items, "Pick Normal Side Color" and "Pick the Other Side Color". After selecting one of them you are presented a standard color picking dialog; the color you select will be used to draw immediately the respective side of the surface. Each time any of those two dialogs opens, the default color is set to the initial color; so that "accepting" without changes means to return to the default color for that side of the surface.

5. Geometry

As you move the mouse, you will see some geometric information about the point signaled currently by the mouse. In all cases we have the parameters, the coordinates of its image (surfaces) and the curvature K, that means scalar curvature (pseudo-Riemannian) or Gauss curvature (surfaces). When the point is pseudo-Riemannian proper, then the value of K is preceded by a bullet. For surfaces we have in addition the mean curvature H, and the two principal curvatures k_1 and k_2.

In the menu "Geometry" only an item can be active at a given moment. By default, you have nothing selected. If your selection is "Coordinate Lines", when you click on the surface in the drawing panel or in the parameter panel, **Superficies** draws the coordinate lines that pass by that point (pseudo-Riemannian or parametric), or the intersection of the surface with the three coordinate planes that pass by that point (implicit).

Below it, you will find the menu item "Light Rays". For surfaces it only works when the "Minkowski option" is set. Select "Light Rays". As you move the mouse, if there are light rays passing by the point, then **Superficies** draws two little segments that represent the directions of null length of the tangent plane. When you click the mouse in one of those points, **Superficies** draws the light rays that pass by that point.

It works in like manner when one of the items "Curvature Lines" or "Asymptotic Lines" is selected (only for surfaces). As the mouse moves, you will see the principal directions or the asymptotic directions at the point signaled by the mouse, if there are any. Preferred principal directions exist only on points that are not umbilical and asymptotic directions only on points with non-positive Gauss curvature.

In some cases, those lines are closed, so that they would go on for ever on the same track; thus, the program strives to detect that and stops them when they reach the initial point. If it fails, "Stop" them and this will stop the branch that is being drawn currently (there are four branches to be drawn from each point).

Next you have several options related to geodesics. All work the same in the four types of surfaces. Suppose that you have selected one of them, say "Exponential Map". Click and drag the mouse; you will see a segment starting at the point that you clicked first and that sticks to the mouse; it is meant to fix the initial tangent of the geodesic that you will get when you release the mouse. If the surface is compact, the geodesic can sometimes go on for ever, and you will need to click the "Stop" button.

In like manner you can get the "Exponential Map", that is the result of applying the exponential map to the segment that sticks to the mouse: it will be the trace of a geodesic with that segment as initial condition and whose length is the length of the segment.

"Geodesic between Points" will attempt to find and draw a geodesic that joins the points of the surface signaled by the ends of the sticking segment and remains in the domain or range of the surface. Note that that geodesic does not always exist, or even if it does, **Superficies** may be unable to find it; anyway, if **Superficies** finds one it is not meant to minimize distance.

Suppose that you draw all geodesics that start from a given point stopping them when they have run for the same fixed length. The ends of those geodesics form the "Geodesic Ball" centered at the initial point and with radius that fixed length. In **Superficies** you will fix the initial point clicking the mouse and the radius by dragging it; the radius will be the length of the sticking segment. When the point is Minkowskian, **Superficies** draws the ball with squared radius equal to the given one, and also the ball with squared radius equal to minus the given one.

When you select "Lines of Constant Geod Curv", by clicking and dragging you get the line of constant geodesic curvature with initial point and direction given by the sticking segment, and with a radius of curvature equal to the length of that segment. To change the "hand" of that curvature, that is from curving right to curving left, press the < Alt > key when clicking the mouse for dragging it.

If the window is a parametric surface, a tube or a pseudo-Riemannian plane, then you have the option "Parallel Transport". Click the mouse and drag; a tiny segment pointing to the right is left at the initial point, and you will see in real time the result of the parallel displacement of that segment along the curve that follows the mouse. Note however that in parametric surfaces or tubes, you must click and drag the mouse in the parameter panel.

For surfaces you have two more items, "Clipping" and "Normal Section". "Clipping" allows you to clip the surface with one or more planes; after selecting this item, you will see the surface enclosed in a sphere with a circle drawn in it; the plane of that circle is the clipping plane; if it cuts the surface, the intersection is drawn in white. When the mouse passes near the circle that determines the clipping plane, the circle changes color (from gray to orange); then you can **translate** it with the pressed mouse. When the mouse is not on the clipping circle, if you press the mouse and drag, the whole sphere is rotated, the clipping plane with it: in this manner, the clipping plane may be put in any desired position. The draw button shows the caption "Clip it"; on pressing it, you obtain the surface clipped by the selected plane. There is a hint about the part that will remain and the part that will be clipped: the sphere that surround the surface is drawn in two colors, gray and yellow, according to the side with respect to the clipping plane. The region of the surface in the yellow side will remain; that in the gray one will disappear.

You may repeat the clipping. While there are any clipping planes, the stop button shows the text "No Clips": it is to redraw the surface without any clips at all.

With the "Normal Section" option, you may clip the surface by a normal plane. To do this, click and drag the mouse on the surface. You will see a little red circle in the first point clicked; **Superficies** shows also the curve on which the surface is cut by the plane which is normal to the surface at the first point and contains the tangent vector determined by the first and the current mouse points. On releasing the mouse, the draw button will show the caption "Clip It", as before. On pressing it, the surface is clipped by the last normal section, so that the part to the left of that tangent disappears. This choice of a normal section can also be made in the parameter panel.

If you select any item in the menu "Draw Mesh" (only for surfaces) you will get a coordinate mesh for parametric surfaces, or a mesh built with the intersections of the surface with equally spaced coordinate planes. The number of lines in each direction is the number appearing in the chosen item.

6. Final notes

This program was written in Borland Pascal, with the Delphi 6 compiler. The program, the source code and this manual is released under GNU General Public License. You can freely retrieve **Superficies** and several samples from the following site:

<ftp://topologia.geomet.uv.es/pub/montesin/Superficies_W32>

References

[1] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, Inc, Englewood Cliffs, New Jersey, 1976.
[2] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

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