Boundary behavior of optimal approximants

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Spaces over the disc

Definition

Dirichlet-type space, $D_\alpha$, is:

$$\{ f \in \text{Hol}(\mathbb{D}) : f(z) = \sum_{k \in \mathbb{N}} a_k z^k, \| f \|^2_\alpha = \sum_{k=0}^{\infty} |a_k|^2 (k + 1)\alpha < \infty \}$$
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**Examples**

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**Today focus on these 3 examples:**

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$\alpha = -1$, $A^2 = \text{Hol}(\mathbb{D}) \cap L^2(\mathbb{D})$

$\alpha = 0$, $H^2 = \text{Hol}(\mathbb{D}) \cap L^2(\mathbb{T})$
**Spaces over the disc**

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**Today focus on these 3 examples:**

**Examples**

\[
\begin{align*}
\alpha = -1, & \quad A^2 = Hol(\mathbb{D}) \cap L^2(\mathbb{D}) \\
\alpha = 0, & \quad H^2 = Hol(\mathbb{D}) \cap L^2(\mathbb{T}) \\
\alpha = 1, & \quad D = Hol(\mathbb{D}) \cap \{ A(f(\mathbb{D})) < \infty \}
\end{align*}
\]
The (forward) *shift operator* is bdd:

\[ S : D_\alpha \rightarrow D_\alpha : Sf(z) = zf(z). \]

A closed subspace \( V \) of \( D_\alpha \) is *invariant* if \( SV \subset V \).
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[f]_\alpha (= [f]) = \overline{\text{span}\{z^k f : k = 0, 1, 2, \ldots\}} = \overline{\mathcal{P}f}.
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\( \mathcal{P} \text{ dense } \subset D_\alpha \Rightarrow [1] = D_\alpha. \)
Cyclicity and invariant subspaces

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A function \( f \) is cyclic (in \( D_\alpha \)) if \([f] = D_\alpha\)

\[ \Leftrightarrow \exists \{p_n\}_{n \in \mathbb{N}} \in \mathcal{P} : \|p_n f - 1\|_\alpha \xrightarrow{n \to \infty} 0 \]
Cyclicity and invariant subspaces

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A closed subspace *V* of *D_\alpha* is *invariant* if *SV \subset V*.

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A function *f* is *cyclic* (in *D_\alpha*) if \([f] = D_\alpha\)

\[ \iff \exists \{p_n\}_{n \in \mathbb{N}} \in \mathcal{P} : \|p_n f - 1\|_\alpha \xrightarrow{n \to \infty} 0 \Rightarrow p_n \to 1/f \text{ pw in } \mathbb{D}. \]
Examples and classical results

- $Z(f) \cap \overline{D} = \emptyset \Rightarrow f \in \text{Hol}(\overline{D}) \Rightarrow f \text{ cyclic in } D_\alpha \Rightarrow Z(f) \cap D = \emptyset$.

Smirnov ('30s): $H_2$ functions factorize as inner $\times$ outer.

Theorem (Beurling, '49)
For $H_2(\alpha = 0)$, cyclic $\iff$ outer. Invariant subspaces generated by a single inner function.

In other spaces, much known but still to be understood.
Examples and classical results

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**Examples and classical results**

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Optimizational viewpoint

- BCLSS (JdAM,’15) and FMS (CMFT,’14): How cyclic is a function?

If we fix $\deg p \leq n$, how fast can $\|p_n f - 1\|_2 \to 0$?

Optimizational viewpoint:

$\Pi_n \text{ort. proj } \Pi_n : D_{\alpha} \to V_n = \{p f : p \in P_n\}$.

$\exists! \Pi_n (1)$, best approximation to $1$ in $V_n$.

Definition: The best approximant to $1/f$ of degree $n$ is the $p_n^{\ast} \in P_n$:

$p_n^{\ast} f = \Pi_n (1)$.

Now, cyclic $\iff \|p_n^{\ast} f - 1\|_2 \to 0 \iff p_n^{\ast}(0) \to 1/f(0)$.

BFKSS: When $f$ not cyclic, $p_n f \to I_I (0)$, $I_I$ "inner part of $f$".
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**Optimizational viewpoint:** $\Pi_n$ orth. proj

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The *best approximant to $1/f$ of degree $n$* is the $p_n^* \in \mathcal{P} : p_n^* f = \Pi_n(1)$. 
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- BCLSS (JdAM,’15) and FMS (CMFT,’14):
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The best approximant to $1/f$ of degree $n$ is the $p^*_n \in P : p^*_n f = \Pi_n(1)$.

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**Definition**
The *best approximant to \( 1/f \) of degree \( n \) is the \( p_n^* \in \mathcal{P} : p_n^* f = \Pi_n(1) \).*

Now, cyclic \( \iff \| p_n^* f - 1 \|_2^2 \to 0 \iff p_n^*(0) \to 1/f(0) \)

BFKSS: When \( f \) not cyclic, \( p_n f \to \overline{l(0)}, \overline{l} \) “inner part of \( f \)”.
We solved these optimization problems:

**Theorem (BCLSS, JdAM’15; FMS, CMFT’14)**

\[ p^*_n(z) = \sum_{j=0}^{n} c_j z^j \] only solution to \( Mc = b \) where

\[ c = (c_j)_{j=0}^{n}, \quad M_{j,k} = \langle z^j f, z^k f \rangle_\alpha, \quad b_k = \langle 1, z^k f \rangle_\alpha. \]
Later we discovered a relation with OPs: Let $\phi_j$ of degree $j$ defined by:

$$\langle \phi_j f, \phi_k f \rangle_\omega = \delta_{j,k},$$

and such that $\hat{\phi}_j(j) > 0$. 

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and such that $\hat{\phi}_j(j) > 0$.

Then we can obtain $\phi_j$ from $p_j$ and $p_{j-1}$ since:

**Theorem (BKLSS, JLMS’16)**

$$p_n(z) = f(0) \sum_{k=0}^{n} \phi_k(0) \phi_k(z)$$
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- Can we obtain a closed formula for $p_n$ in terms of a closed formula for $f$?
- Can we find $p_n$ faster than inverting $M$ for each $n$?

YES, if $f$ is polynomial.
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YES, if $f$ polynomial.
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$$g \perp z^t(2 - 3z + z^2) \quad t = 0, \ldots, n$$
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- Additional restrictions:

\[
(1 - p_n f)(1) = (1 - p_n f)(2) = 1
\]
A general closed formula

**Theorem**

\[ \exists A_n = (A_{1,n}, \ldots, A_{d,n})^* \quad (\text{ind. of } k): \text{ for } k = 0, \ldots, n + d, \]

\[
d_{k,n} = \frac{1}{\omega_k} \sum_{i=1}^{d} A_{i,n} z_i^k.
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where

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\[ E_{Z,n,l,m} = \sum_{k=0}^{n+d} \frac{z_m^k z_l^k}{\omega_k}. \] \quad (3)

So inverting a \( d \times d \) matrix we can obtain a closed formula for all \( n \). Also, for \( p_n \) and hence for \( \phi_k \).
Corollary

\[ \text{dist}^2(1, \mathcal{P}_n f) = -\sum_{i=1}^{d} A_{i,n} = v_0 E_{Z,n}^{-1} v_0^*. \]

In particular,

\[ \sum_{i=1}^{d} A_{i,n} \in [-1, 0]. \]

Also, if \( Z(f) \subset \mathbb{D} \), then

\[ \text{dist}^2(1, [f]) = v_0 K_Z^{-1} v_0^*. \]
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Notice \( E_{Z,\infty,l,m} = k_{zm}(z_l). \)
Wiener norm

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**Theorem**

Let \( f \in \mathcal{P} : Z(f) \cap D = \emptyset \). \( \exists C \in \mathbb{R} : \forall n \in \mathbb{N}, \)

$$\|p_nf - 1\|_A \leq C.$$
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Hence, unif. bounded.
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Hence, unif. bounded.
Perhaps, true if \( f \in A(\mathbb{T})? \)
Let $Z(f) \cap \mathbb{D} = \emptyset$, $z_0 \in \overline{\mathbb{D}} \setminus Z(f)$. Then

$$(p_n f - 1)(z_0) \to 0 \quad \text{as} \quad n \to \infty.$$ 

Convergence is locally uniform.
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*Convergence is locally uniform.*

So polynomials are “well behaved” on the boundary... Are there “badly behaved” functions?
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To be continued...
Coming up work BMS and Ivrii