Hence, for all $x, y \in GL_2(\mathbf{R})$, $\mathrm{trace}([x,y]^2) \geqq -2$. Is it true that for any integers $p_1, \ldots, p_m, q_1, \ldots, q_m$ as above the set

$$\mathrm{trace}(x^{p_1}y^{q_1}\cdots x^{p_m}y^{q_m}) : x, y \in GL_2(\mathbf{R})\}$$

includes the interval $[-2,\infty)$?

3. For extensive studies of $GL_2(\mathbf{C})$ and related groups see [1] and [2].

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CLASSROOM NOTES

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PICARD'S THEOREM WITHOUT TEARS

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1. Recently I noticed an approach to a restricted version of the Little Picard Theorem so short and simple that it can be presented to a class of undergraduates in a single lecture. This approach leads naturally to important generalizations of Liouville's Theorem (in which a one-sided bound on the real or imaginary part of a function replaces a bound on the modulus of the function); moreover, it exhibits Picard's theorem clearly as an analogue and extension of the Fundamental Theorem of Algebra. In fact, the proof is by an induction, the initial step of which is just the Fundamental Theorem of Algebra.

While it seems to me unlikely that this approach is really new, I have been unable to find it in the literature; and discussion with knowledgeable colleagues has failed to turn up a reference.

2. To begin with, we need some notation. Let $F(z) = U(z) + iV(z)$ be an entire function; it is understood that U and V are real functions. We set

$$M(r,F) = \max_{|z|=r} |F(z)| \qquad A(r,F) = \max_{|z|=r} U(z);$$

when no confusion can arise, we simply write $M(r)$, $A(r)$. The n-times iterated logarithm, $\log_n$, is defined by $\log_1 t = \log t$, $\log_n t = \log(\log_{n-1} t)$, where all logarithms are to the base e. Similarly, we denote by $\exp_n$ the n-times iterated exponential.

Our interest centers on the class of entire functions $F(z)$ for which

$$\limsup_{r\to\infty} \frac{\log_n M(r)}{\log r} < \infty \tag{1}$$

for some positive integer n. Among the functions satisfying (1) are the exponentials $\exp_{n-1} z^m$, m a positive integer. On the other hand, not every entire function satisfies (1) for some n. Indeed, if the numbers $a_k > 0$ are chosen appropriately small, the series $\Sigma a_k \exp_k z$ converges uniformly on each disc about the origin and hence defines an entire function $F(z)$. Since $M(r) > a_n \exp_n r$ for each n, it is clear that F cannot satisfy (1).

While functions which satisfy (1) do not exhaust all entire functions, I know of no function or class of entire functions of any interest in complex analysis that does not already satisfy (1) for a small value of n. The case $n = 2$ is, of course, the much studied class of functions of *finite order.*

We shall prove the following result, apparently midway between the Big and Little Theorems of Picard. (Actually, it is an immediate consequence, hardly ever drawn, of the Little Theorem.)

THEOREM. *Let $F(z)$ be a nonconstant entire function satisfying* (1) *for some n. If F fails to take on some value $a \in \mathbf{C}$, it takes on every other value $b \in \mathbf{C}$ $(b \neq a)$ infinitely often.*

Thus, either F takes on *every* value in $\mathbf{C}$ or it omits a single value and takes on every other value infinitely often.

3. For the proof, we need the following result, which bounds $M(r)$ in terms of $A(R)$, $R > r$.

BOREL–CARATHÉODORY INEQUALITY. *Let $0 \leqq r < R$. Then*

$$M(r) \leqq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |F(0)|. \tag{2}$$

Proof. If $F(z) = \Sigma_{n=0}^{\infty} a_n z^n$, where $a_n = \alpha_n + i\beta_n$ (α_n, β_n real), we have

$$U(Re^{i\theta}) = \operatorname{Re} \sum_{n=0}^{\infty} (\alpha_n + i\beta_n) R^n (\cos n\theta + i \sin n\theta)$$

$$= \sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) R^n,$$

where the series converges uniformly in θ. For $n \geqq 1$ we have

$$\pi \alpha_n R^n = \int_0^{2\pi} U(Re^{i\theta}) \cos n\theta \, d\theta$$

$$\pi \beta_n R^n = -\int_0^{2\pi} U(Re^{i\theta}) \sin n\theta \, d\theta,$$

so that

$$\pi a_n R^n = \int_0^{2\pi} U(Re^{i\theta}) e^{-in\theta} d\theta = \int_0^{2\pi} [U(Re^{i\theta}) - A(R)] e^{-in\theta} d\theta.$$

Thus

$$\pi |a_n| R^n \leqq \int_0^{2\pi} |U(Re^{i\theta}) - A(R)| \, d\theta = \int_0^{2\pi} [A(R) - U(Re^{i\theta})] \, d\theta = 2\pi [A(R) - \alpha_0],$$

so that

$$|a_n| R^n \leqq 2[A(R) + |F(0)|] \tag{3}$$

and $|a_n| r^n \leqq 2[A(R) + |F(0)|](r/R)^n$ for $n \geqq 1$. It follows that

$$|F(re^{i\theta}) - F(0)| \leqq \sum_{n=1}^{\infty} |a_n| r^n$$

$$\leqq 2[A(R) + |F(0)|] \sum_{n=1}^{\infty} (r/R)^n$$

$$= \frac{2r}{R-r} A(R) + \frac{2r}{R-r} |F(0)|;$$

hence

$$|F(re^{i\theta})| \leqq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |F(0)|,$$

as required.

An immediate corollary is a version of Liouville's theorem which, though well known to experts, too often escapes mention in a first course in function theory.

LIOUVILLE'S THEOREM. *Let $F(z) = U(z) + iV(z)$ be entire and suppose that there exist positive constants C, K, and α such that $U(z) \leqq C|z|^{\alpha}$ whenever $|z| \geqq K$. Then $F(z)$ is a polynomial of degree no greater than α.*

Proof. The hypothesis implies that for each integer $n > \alpha$

$$\limsup_{R\to\infty} A(R)/R^n \leqq 0,$$

so by (3) $a_n = 0$ for $n > \alpha$.

Another consequence is the

FUNDAMENTAL THEOREM OF ALGEBRA. *Every nonconstant polynomial has a root in* **C**.

Proof. Suppose the polynomial $p(z)$ never vanishes. Then $p(z) = e^{F(z)}$ for some entire function F and $|p(z)| = \exp\{\operatorname{Re} F(z)\}$. Thus $M(r,p) = e^{A(r,F)}$, so that

$$\limsup_{r\to\infty} \frac{A(r,F)}{\log r} = \limsup_{r\to\infty} \frac{\log M(r,p)}{\log r} < \infty.$$

It follows that for large r we have

$$A(r,F) \leqq C(\log r) \leqq r^{1/2},$$

so by Liouville's Theorem F is constant. Hence $p = e^F$ is constant.

The preceding proof fits nicely into the present circle of ideas. A much simpler argument, which uses only a special case of the Cauchy integral formula, can also be given: if $p(z)$ never vanishes, $q(z) = 1/p(z)$ is entire and $q(0) = 1/p(0) \neq 0$. But

$$q(0) = \frac{1}{2\pi i} \int_{|z|=R} q(z)\,dz/z = \frac{1}{2\pi} \int_0^{2\pi} q(Re^{i\theta})\,d\theta$$

so that $|q(0)| \leqq M(R,q)$. It is easy to see that if p is nonconstant this last quantity tends to 0 as R tends to infinity; choosing R large enough gives the required contradiction.

We shall use the inequality of Borel and Carathéodory in the following form:

LEMMA. *Let F be a nonconstant entire function. Then $M(r) \leqq 3A(2r)$ for $r > r_0(F)$.*

Proof. Take $R = 2r$ in (2) to get $M(r) \leqq 2A(2r) + 3|F(0)|$. Since F is nonconstant, Liouville's theorem shows that $A(2r) \to \infty$. Done.

4. We now prove the theorem. For $n = 1$, condition (1) implies that $|F(z)| \leqq |z|^m$ for some integer $m > 0$ and all $|z| \geqq 2$. Thus F is a polynomial, so, by the Fundamental Theorem of Algebra (applied to $F(z) - a$), F takes on each value $a \in \mathbf{C}$.

Suppose the result has been proved for $n = k$ and let F satisfy (1) with $n = k + 1$. If F fails to take on the value a, we have $F(z) - a = e^{G(z)}$, where $G(z)$ is again entire. Clearly $F(z) - a$ still satisfies (1) with $n = k+1$, so the relation $M(r, e^G) = e^{A(r,G)}$ yields

$$\limsup_{r\to\infty} \frac{\log_k A(r, G)}{\log r} < \infty .$$

This inequality remains valid if $A(r, G)$ is replaced by $3A(2r, G)$. Thus, applying the lemma to G, we obtain

$$\limsup_{r\to\infty} \frac{\log_k M(r, G)}{\log r} \leqq \limsup_{r\to\infty} \frac{\log_k 3A(2r, G)}{\log r} < \infty .$$

By the induction hypothesis, G takes on every value in **C** with at most one exception. In particular, for each fixed value $w \in \mathbf{C}$, G takes on all values $w + 2\pi i n$, $n = 0, \pm 1, \pm 2, \ldots$, with at most one exception. It is now obvious that $F(z) = e^{G(z)} + a$ takes on each value in $\mathbf{C}\backslash\{a\}$ infinitely often.

5. The elegant proof of the Borel–Carathéodory inequality given above is taken almost verbatim from [**3**, p. 16]; it is a refinement of an argument which goes back at least to Hadamard [**4**]. Quite a different proof, involving Schwarz' lemma, can be found in [**5**, pp. 174–5]. Borel used his inequality to give the first "elementary" proof of the Little Picard Theorem [**1**]. His proof and the argument given here share only conventional elements: their central mechanisms seem quite different.

Finally, while we are very far from claiming that our result provides a satisfactory substitute for the full Picard theorem, we find it both amusing and instructive to consider Borel's own remarks [**2**, pp. 145–6] on generality in function theory with this (or, for that matter, any other) question in mind.

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A NOTE ON ASYMPTOTIC EXPANSIONS

KUSUM SONI

One of the simplest techniques, used in the asymptotic expansion of a function defined by a definite integral, is integration by parts. If this technique can be used, the successive terms in the expansion are obtained by repeated integration by parts. However, we obtain an asymptotic series only if the order of the remainder after n terms is lower than the order of the nth term for every positive integer n. In most of the examples we encounter in the literature, this condition is satisfied. Nevertheless, it should not be taken for granted. The following simple example is helpful in emphasizing this point. Let

$$I(x) = \int_0^1 t^\alpha J_0(xt)\,dt, \qquad -1 < \alpha < \tfrac{1}{2}.$$

$J_m(t)$ is the Bessel function of the first kind of order m. Since $(t^m J_m(t))' = t^m J_{m-1}(t)$, we can use integration by parts n times to obtain