Problemas para el verano 2018

1) Consider the polynomial $f(X) := X^2 + a$, with $a \in \mathbb{Z}$. Define the following sequence:

$$\begin{aligned} c_1 &:= -a; \\ c_n &:= f(c_{n-1}), \quad \forall n > 1. \end{aligned}$$

prove that if $-a$ is not a square, then c_n is never a square.

2) Find all integral solutions to the equation:

$$y^2 = x^3 + 7$$

3) Find all integral solutions to the equation:

$$y^2 = x^3 - 2$$

(if you need it, you can use the fact that the ring $\mathbb{Z}[\sqrt{-2}]$ has unique factorization, hence the gcd works in the same as it does over $\mathbb{Z}$).