Prueba de Selección
7 de marzo de 2024, de 17:00 a 20:00
Soluciones

Apellidos, Nombre, curso, grado, email ______________________________

1) On the plane, $n \geq 5$ points are given, such that no three of them are collinear. Prove that one can find at least $\binom{n-3}{2}$ different convex quadrilaterals with vertices at these points.

Sol.: *Given two points $F \neq G$, we denote by FG the straight line passing through these points and by $[FG]$ the segment of this line with endpoints in F and G.*

Fix three points A, B, C from the given set that are vertices of the convex hull of this set. One can choose $\binom{n-3}{2}$ pairs, which are subsets of the rest of the given points.

The line DE intersects 0 or 2 sides of the triangle ΔABC. We assert that one can choose a side s of ΔABC so that DE does not intersect s and

(*) *the points D, E, and the two endpoints of s are vertices of a convex quadrilateral.*

This will imply the statement of the problem.

Denote by ℓ the straight line containing s. We will use the following

Fact: *if the line DE does not intersect the segment s and the points D, E are on the same side of ℓ, then D, E, and the two endpoints of s are vertices of a convex quadrilateral.*

To see (*), notice first that the lines AB, BC and AC divide the plain into 7 regions. Three of these regions are infinite angles. By our choice of A, B, C, the points D and E do not belong to these angles (and do not lie on the lines AB, BC and AC). So D, E belong to the four resting regions. One of them is the triangle ABC and the other three, which we denote as W_1, W_2 and W_3, are infinite.

We consider the following cases.

<u>*Case 1:*</u> *D, E belong to the same region. Then for any choice of s as above, D, E are on the same side of ℓ, and by the above Fact, (*) holds.*

<u>*Case 2:*</u> *One of the points D, E belongs to the triangle ABC, and the other point to an infinite region, say, W_1. Choose s in any way as above. The union of the closures of ΔABC and W_1 is an angle U, and the intersection of their boundaries is one of the sides of the triangle, say, t. Since t separates the angle U into two connected components, and D, E lie in different components, the segment $[DE]$ crosses t. Hence ℓ contains one of the sides of the angle U, and therefore D, E lie on the same side of ℓ. The above Fact gives (*).*

<u>*Case 3:*</u> *D, E belong to two different infinite regions, say, W_1 and W_2. We can assume that C belongs to both boundaries ∂W_1, ∂W_2, A is a vertex of W_1 and B is a vertex of W_2. Then the intersection of the line AB with each of the boundaries ∂W_1, ∂W_2 is a half-line. To fix notation, we assume that $D \in W_1$, $E \in W_2$.*

Since D, E are on the same side of AB, $[DE]$ does not intersect $[AB]$. We set $s = [AB]$. If s is parallel to DE, A, B, D, E are vertices of a convex quadrilateral. If not, denote by P the intersection point of the lines AB and DE, so that D, E and P are on the same line.

Suppose $P \in s$. It cannot happen that $D \in [PE]$, for in this case E, P are on the same side of the line AC and D is on the other side. Similarly, $E \notin [PD]$. It has been seen already that $P \notin [DE]$.

This contradiction shows that in reality $P \notin s$. Hence the above Fact yields that A, B, D, E are vertices of a convex quadrilateral.

Since these are all possible cases, the proof is concluded.

Remark: *Now I know a little bit simpler argument, choosing A, B, C to be three consecutive vertices of the convex hull of the n given points. - Dmitri*

2) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $f(x) > 0$ for $0 < x \leq 1$. Prove that there exists a convex function $g : [0,1] \to \mathbb{R}$ such that $g(0) = 0$ and $0 < g(x) \leq f(x)$ for $0 < x \leq 1$.

Sol.: *First notice that, whenever $\{h_t\}_{t \in T}$ is a family of convex functions on $[0,1]$ such that*

$$g(x) := \sup_{t \in T} h_t(x)$$

is finite for all $x \in [0,1]$, the function g is convex. (Here T is an index set.) Indeed, for any $s \in (0,1)$, any $x, y \in [0,1]$ and any $t \in T$,

$$h_t(sx + (1-s)y) \leq s h_t(x) + (1-s) h_t(y) \leq s g(x) + (1-s) g(y).$$

Taking the supremum over t in the left hand side, we get

$$g(sx + (1-s)y) \leq s g(x) + (1-s) g(y),$$

which shows that g is convex on $[0,1]$. The required function g will be constructed in this way. Namely, for $t \in T := (0,1]$, we define

$$h_t(0) = h_t(t) = 0, \quad h_t(1) = \min_{x \in [t,1]} f(x) > 0,$$

and continue the function h_t to $[0,1]$ as a linear function separately on the intervals $[0,t]$ and $[t,1]$. Then for each $t \in (0,1]$, h_t is convex and $0 \leq h_t \leq f$ on $[0,1]$. Next put $g(x) = \sup_{0<t\leq 1} h_t(x)$. Then g is convex and $0 \leq g \leq f$ on $[0,1]$. Since $h_t(x) > 0$ whenever $0 < t < x$, it follows that $g > 0$ on $(0,1]$. So g satisfies all the required properties.

Remark *It is known that any convex function, defined on some interval, is continuous at any point except possibly at the endpoints of this interval. It is clear that g is continuous at 0. Since g is convex, $g(0) = 0$ and $g > 0$ on $(0,1]$, it follows that g increases on $[0,1]$. So, if one redefines the value $g(1)$ as $\lim_{x \to 1} g(x)$, one gets a continuous convex function, satisfying all the requirements.*

Sketch of Sol 2: *Put $h(x) = \min_{t \in [x,1]} f(t)$. Then h is an increasing function and is continuous (it has to be proved), and $0 < h \leq f$ on $(0,1]$. Therefore $g(x) := \int_0^x h(t)\,dt$ is convex. It is easy to check that $0 < g \leq h \leq f$ on $(0,1]$.*

3) Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

Sol.: *Suppose on the contrary that the set B of values of n for which Bob has a winning strategy is finite; for convenience, we include $n = 0$ in B, and write $B = \{b_1, \dots, b_m\}$. Then for every nonnegative integer n not in B, Alice must have some move on a heap of n stones leading to a position in which the second player wins. That is, every nonnegative integer not in B can be written as $b + p - 1$ for some $b \in B$ and some prime p. However, there are numerous ways to show that this cannot happen.*

First solution: *Let t be any integer bigger than all of the $b \in B$. Then it is easy to write down t consecutive composite integers, e.g., $(t+1)! + 2, \dots, (t+1)! + t + 1$. Take $n = (t+1)! + t$; then for each $b \in B$, $n - b + 1$ is one of the composite integers we just wrote down.*

Second solution: *Let $p_1, \dots, p_{2m}$ be any prime numbers; then by the Chinese remainder theorem, there exists a positive integer x such that*

$$\begin{aligned} x - b_1 &\equiv -1 \pmod{p_1 p_{m+1}} \\ &\cdots \\ x - b_n &\equiv -1 \pmod{p_m p_{2m}}. \end{aligned}$$

For each $b \in B$, the unique integer p such that $x = b + p - 1$ is divisible by at least two primes, and so cannot itself be prime.

Third solution: *Put $b_1 = 0$, and take $n = (b_2 - 1) \cdots (b_m - 1)$; then n is composite because $3, 8 \in B$, and for any nonzero $b \in B$, $n - b_i + 1$ is divisible by but not equal to $b_i - 1$. (One could also take $n = b_2 \cdots b_m - 1$, so that $n - b_i + 1$ is divisible by b_i.)*

4) Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots \right\}.$$

Sol.: *It suffices to prove that for any relatively prime positive integers r, s, there exists an integer n with $a_n = r$ and $a_{n+1} = s$. We prove this by induction on $r + s$, the case $r + s = 2$ following from the fact that $a_0 = a_1 = 1$. Given r and s not both 1 with $\gcd(r, s) = 1$, we must have $r \neq s$. If $r > s$, then by the induction hypothesis we have $a_n = r - s$ and $a_{n+1} = s$ for some n; then $a_{2n+2} = r$ and $a_{2n+3} = s$. If $r < s$, then we have $a_n = r$ and $a_{n+1} = s - r$ for some n; then $a_{2n+1} = r$ and $a_{2n+2} = s$.*