Hoja 6

1) Show that if n has $p-1$ digits all equal to 1, where p is a prime not equal to $2, 3$ or 3, then n is divisible by p.

2) Determine all positive integers n for which there exist positive integers a, b and c satisfying

$$2a^n + 3b^n = 4c^n.$$

3) Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

4) Consider a planar region of area 1, obtained as the union of fintely many disks. Prove that from these disks we can select some that are mutually disjoint and have total area at least $\frac{1}{9}$.

5) Let n be a positive integer and let $a_1, \ldots, a_k$ $(k \geq 2)$ be distinct integers from the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

6) Find all the functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(xf\left(\frac{1}{y}\right)\right) = xf\left(\frac{1}{x+y}\right),$$

for all $x, y > 0$.

7) Denote by $[x]$ the entire part of x. Prove that for any natural number n, the integer $[(2+\sqrt{3})^n]$ is odd.

8) Let $u(x)$ be a positive continuous function on $[0, +\infty)$, which satisfies

$$\int_0^\infty \frac{dx}{u(x)} < \infty.$$

Prove that

$$\lim_{A \to +\infty} \frac{1}{A^2} \int_0^A u(x)\, dx = \infty.$$