Hoja 6

1) A language has an alphabet consisting of n letters. Each word of this language has no more than m letters, and no one of the words is a beginning of another one. (It is understood that the beginnings of a word $L_1L_2\dots L_k$, where $L_1, L_2, \dots, L_k$ are letters of the alphabet, are words of the form $L_1L_2\dots L_s$, where $1 \le s \le k$.)

Show that

$$\sum_{k=1}^{m} \frac{a_k}{n^k} \le 1;$$

here a_k is the number of words of the language that have exactly k letters.

2) Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two sequences of real numbers satisfying $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = +\infty$. Show that there are indices m and n such that $|a_n - a_m| > 1$ and $|b_n - b_m| > 1$.

3) Find al positive integers $n > 2$ such that n divides the number

$$\prod_{\substack{p<q<n,\\ p,q \text{ primes}}} (p+q).$$

4) Let u be a positive continuous function on $[0, +\infty)$ such that $\displaystyle\int_0^\infty \frac{d\,x}{u(x)} < \infty$. Prove that

$$\lim_{A\to\infty} \frac{1}{A^2} \int_0^A u(x)\,dx = \infty.$$